A CAROUSEL PROPERTY FOR COMPACT CONVEX SETS

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Carleton University
Algorithms Seminar

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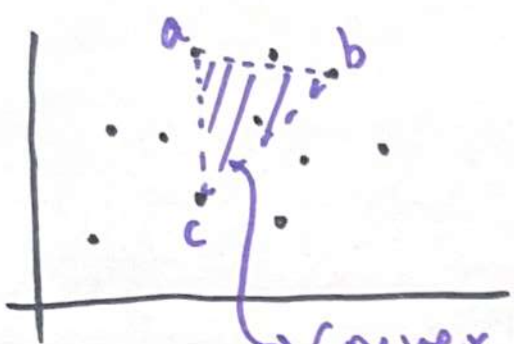
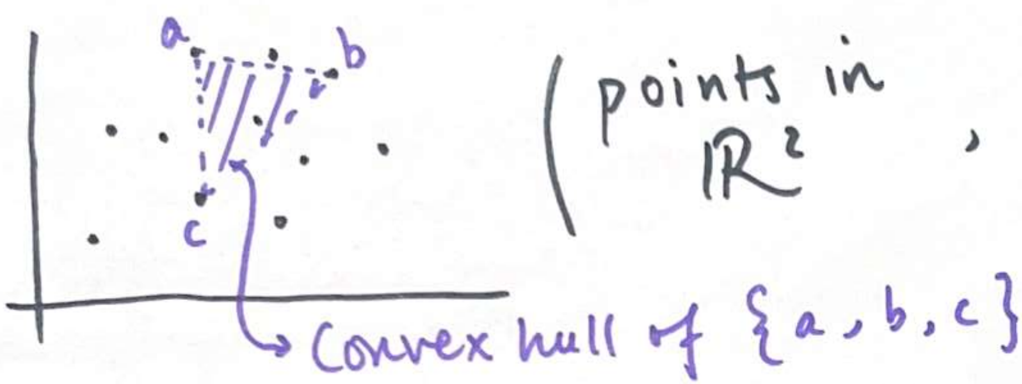
1 CONVEXITY :



(points in $\mathbb{R}^2$, convex hull)

Convex hull of $\{a, b, c\}$

① CONVEXITY :



(points in $\mathbb{R}^2$, convex hull)

More generally, a <u>CONVEX GEOMETRY</u> is a set X and a closure operator $\phi : 2^X \to 2^X$ satisfying:

CLOSURE PROPERTIES
$$\begin{cases} A \subset \phi(A) \\ A \subset B \Rightarrow \phi(A) \subset \phi(B) \\ \phi(A) = \phi(\phi(A)) \end{cases}$$

ANTI-EXCHANGE RULE :
For convex A, $x, y \notin A$,
$x \in \phi(A \cup \{y\})$
$\Rightarrow y \notin \phi(A \cup \{x\})$

Q. Does every convex geometry look like

$$\left\{ \begin{matrix} \text{points in} \\ \mathbb{R}^n \end{matrix} , \ \text{convex hull} \right\} \quad ?$$

$$\Big\updownarrow \cong \ ?$$

$$\left\{ \begin{matrix} \text{a set} \\ X \end{matrix} , \ \begin{matrix} \text{closure operator} \\ \phi : 2^X \longrightarrow 2^X \end{matrix} \right\}$$

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$$\left(\begin{matrix} \text{points in} \\ \mathbb{R}^n \end{matrix} , \text{convex hull} \right) \ ?$$

$$\Big\updownarrow \cong ?$$

$$\left(\begin{matrix} \text{a set} \\ X \end{matrix} , \begin{matrix} \text{closure operator} \\ \phi : 2^X \longrightarrow 2^X \end{matrix} \right)$$

A. NO.

$$\left(\{a, b\}, \phi \right) \xrightarrow{\ \not\cong\ } \left(\begin{matrix} \text{two points} \\ \text{in } \mathbb{R}^n \end{matrix} , \begin{matrix} \text{convex} \\ \text{hull} \end{matrix} \right)$$

$$\boxed{\phi(\{a\}) = \{a, b\}}$$
$$\phi(\{b\}) = \{b\}$$
$$\phi(\{a,b\}) = \{a, b\}$$
$$\phi(\{\}) = \{\}$$

No <u>representation</u> with points in $\mathbb{R}^n$ exists.

A. NO... but possible with CIRCLES

$$\left(\{a, b\}, \phi\right) \longrightarrow \left(\text{circles in } \mathbb{R}^2, \text{ convex hull}\right)$$

$\phi(\{a\}) = \{a, b\}$

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$a \longmapsto$ 

$b \longmapsto$

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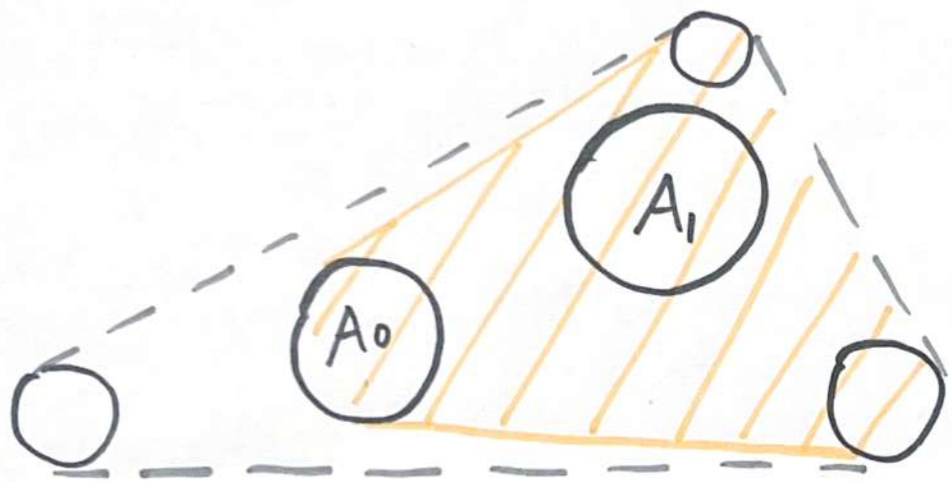
More geometry $\Longrightarrow$ More representations

Restricting to $\mathbb{R}^2$, are circles enough?

NO. (ADARICHEVA-BOLAT, 2019)

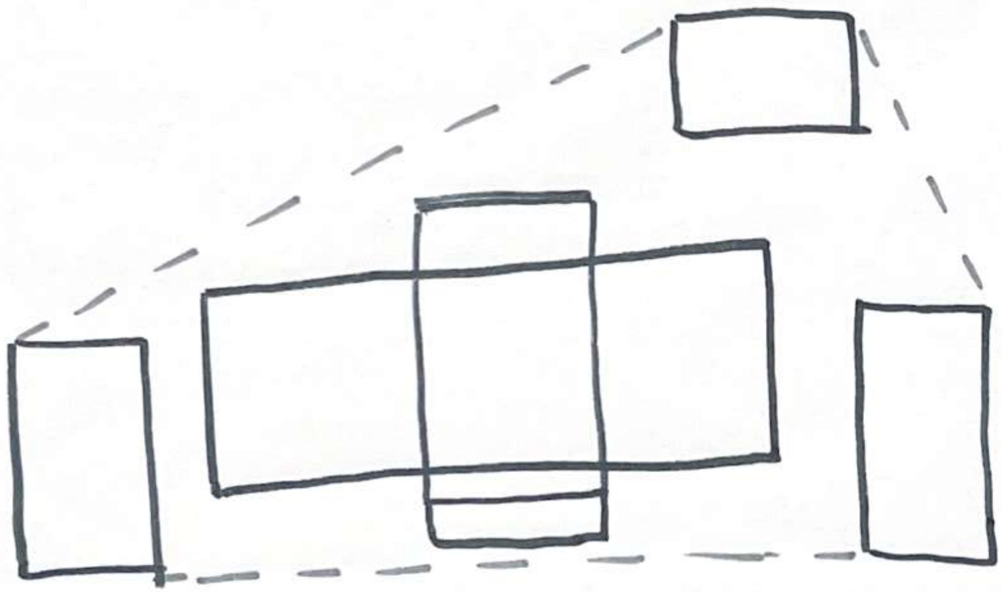


<u>Thm</u>. If A_0, A_1 are disks in $\mathbb{R}^2$ and G is the convex hull of three disks g_1, g_2, g_3, where $A_0, A_1 \subset \mathrm{Conv}(g_1, g_2, g_3)$, then there exist $i \in \{0,1\}$, $j \in \{1,2,3\}$ such that

$$A_i \subset \mathrm{Conv}(A_{1-i}, \{g_1, g_2, g_3\} \setminus \{g_j\})$$

More complex polygons don't have this issue.

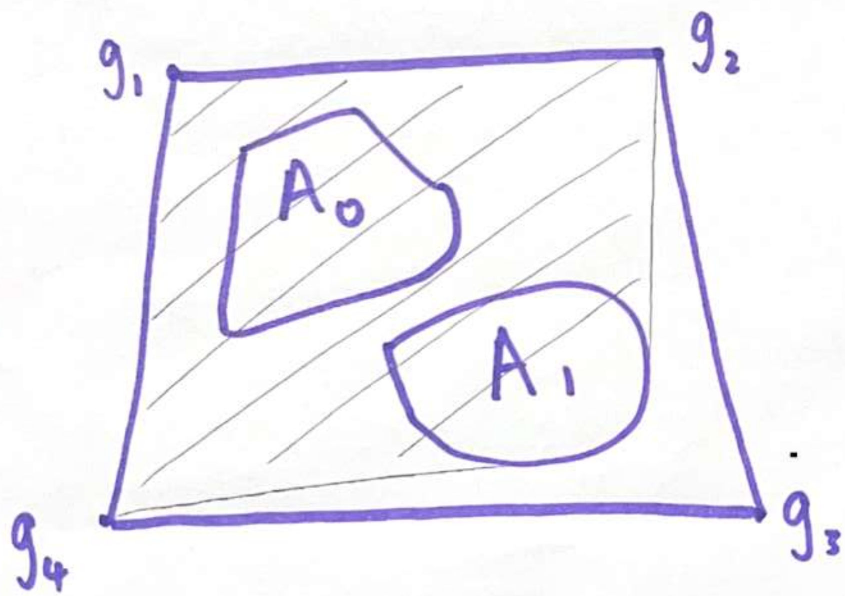
<u>Thm</u> (Richter-Rogers, 2017). Any convex geometry can be represented by n-gons in $\mathbb{R}^2$ for sufficiently large n.



The **WEAK CAROUSEL RULE**: Given $\mathcal{A} = \{A_0, A_1\}$ convex compact subsets of the plane, and $G = \mathrm{conv}(g_1, \dots, g_n)$ a convex n-gon containing $\mathcal{A}$, we say $(\mathcal{A}, G)$ satisfy the WCR if:

$\exists\, i \in \{0,1\},\ j \in \{1, \dots, n\}$ such that

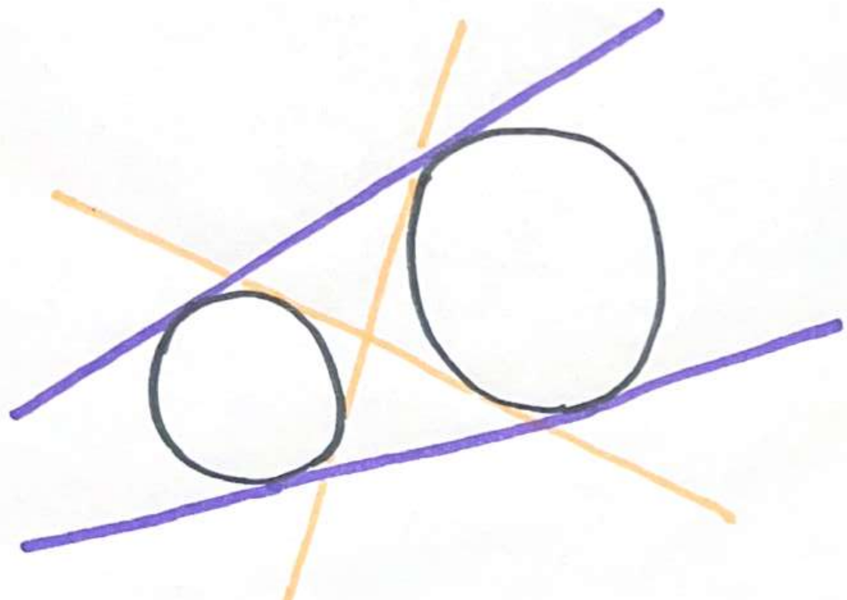
$$A_i \subset \mathrm{conv}\big(A_{1-i},\ \{g_1, \dots, g_n\} \setminus \{g_j\}\big).$$



<u>Thm</u>. (S., 2025). Suppose

$$\#\left\{\begin{array}{l}\text{common supporting} \\ \text{lines of } A_0, A_1\end{array}\right\} < n. \text{ Then}$$

the weak carousel rule holds.

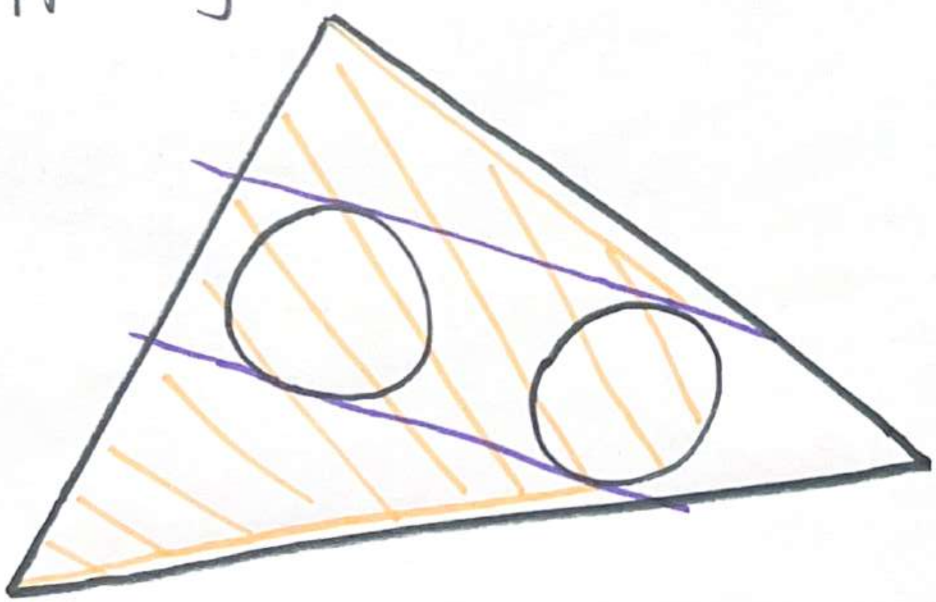


common supporting lines

not

Cor. (Adaricheva – Bolat, 2019). Two disks in a triangle satisfy the weak carousel rule.

Pf. Two Disks have at most two common supporting lines.



Thm (S., 2025) Let $A_0, A_1 \subset \mathbb{R}^2$ be convex and compact. Let $G = \mathrm{conv}(g_1, \dots, g_n)$ be an n-gon containing A_0, A_1. If

$$\#\left\{\begin{matrix}\text{common supporting} \\ \text{lines of } A_0, A_1\end{matrix}\right\} < n,$$

then $\exists\, i \in \{0,1\},\ j \in \{1, \dots, n\}$ such that

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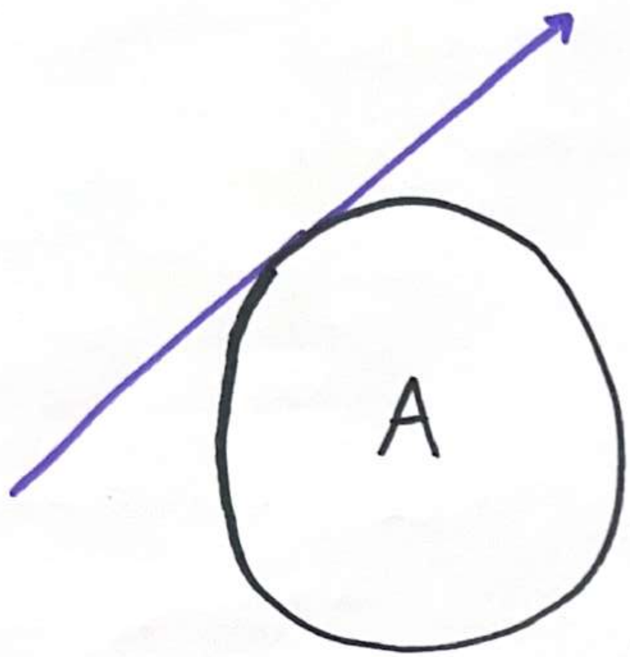
Pf idea: "Slide-turning"

(Czédli–Stachó, 2016)

Slide-turning.



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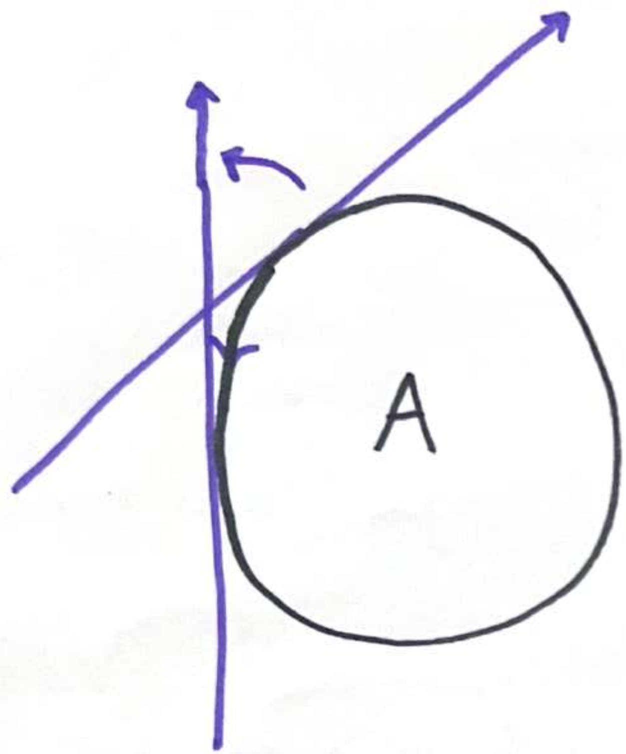

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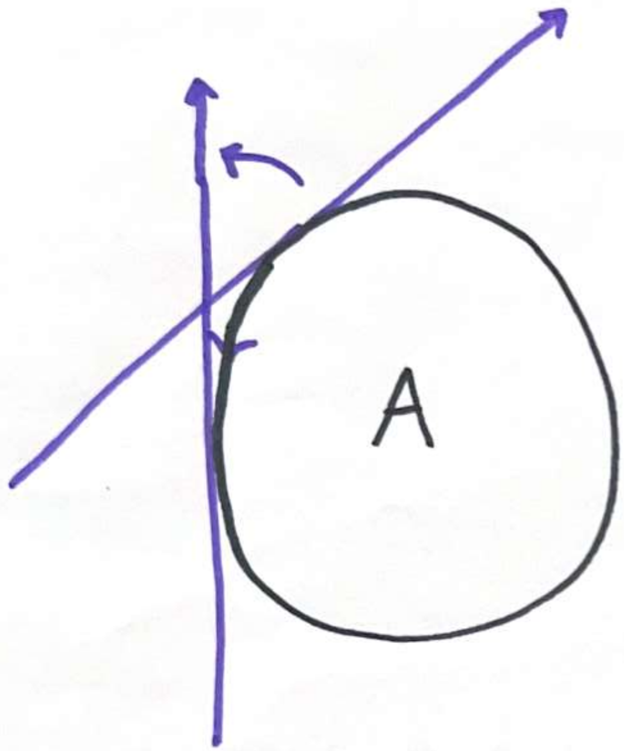

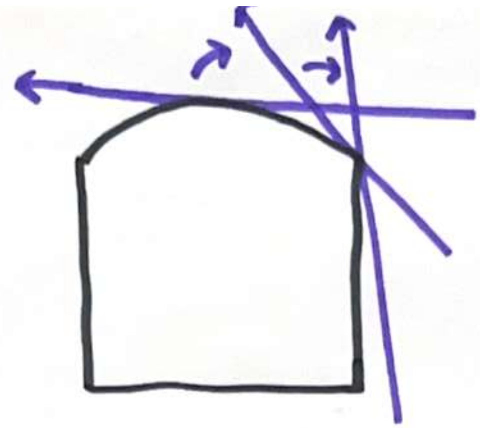

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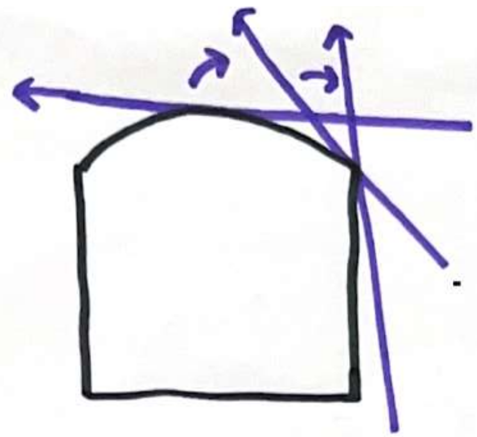

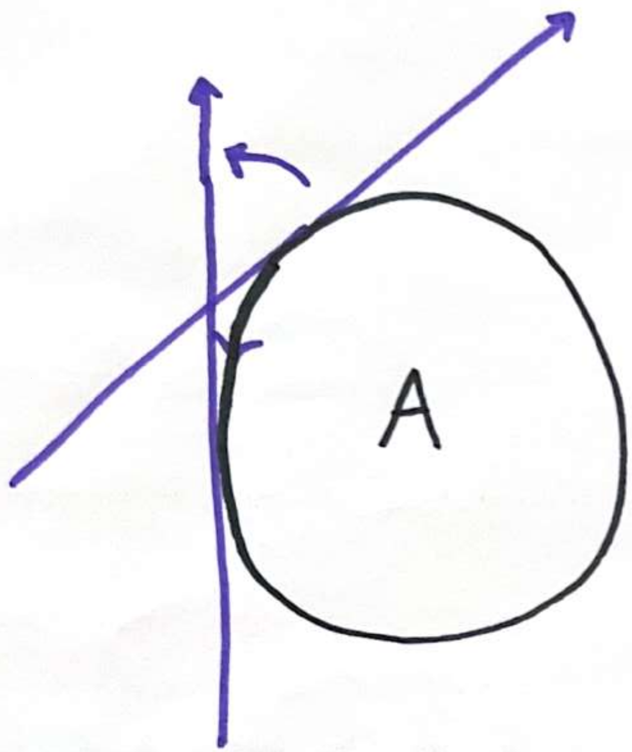

Thm (Czédli – Stachó, 2016).
If A is nonempty, convex, compact, then

$$\left\{ (a, \ell) \;:\; a \in \partial A,\ \ell \text{ is a supporting line of } A,\ a \in \ell \right\}$$

is a simple rectifiable curve in $\mathbb{R}^2 \times S^1 \subseteq \mathbb{R}^4$.

Takeaway: We have a bijection

$$[0, 2\pi) \longleftrightarrow \left\{ \begin{array}{c} \text{oriented supporting} \\ \text{lines of } A \end{array} \right\}$$

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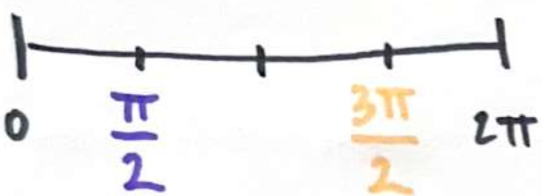

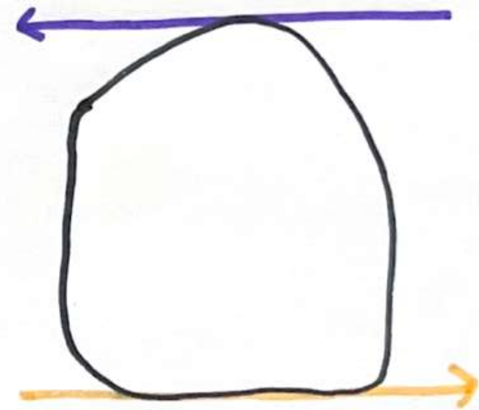

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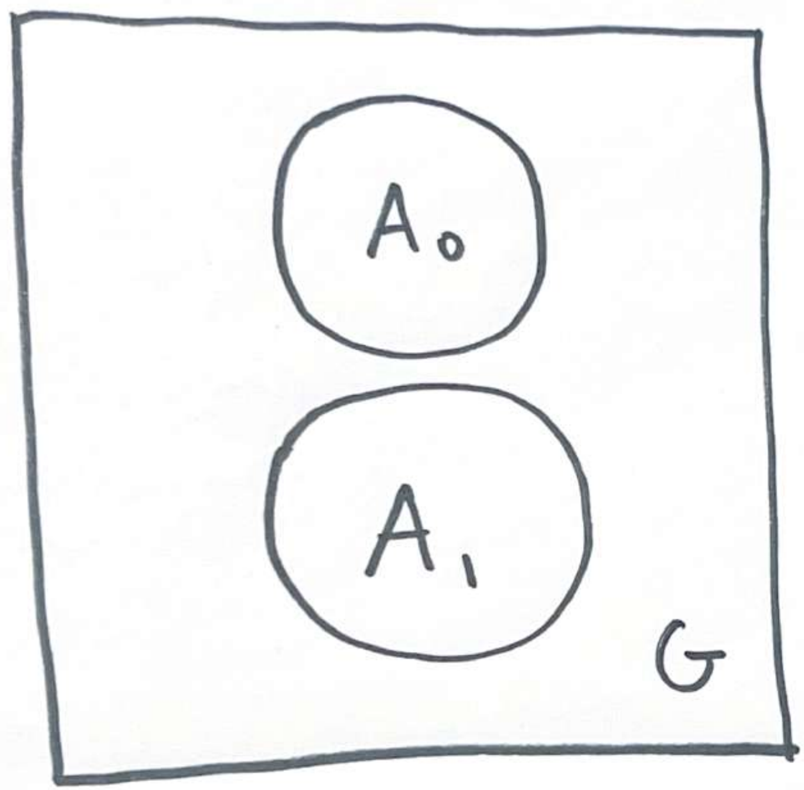

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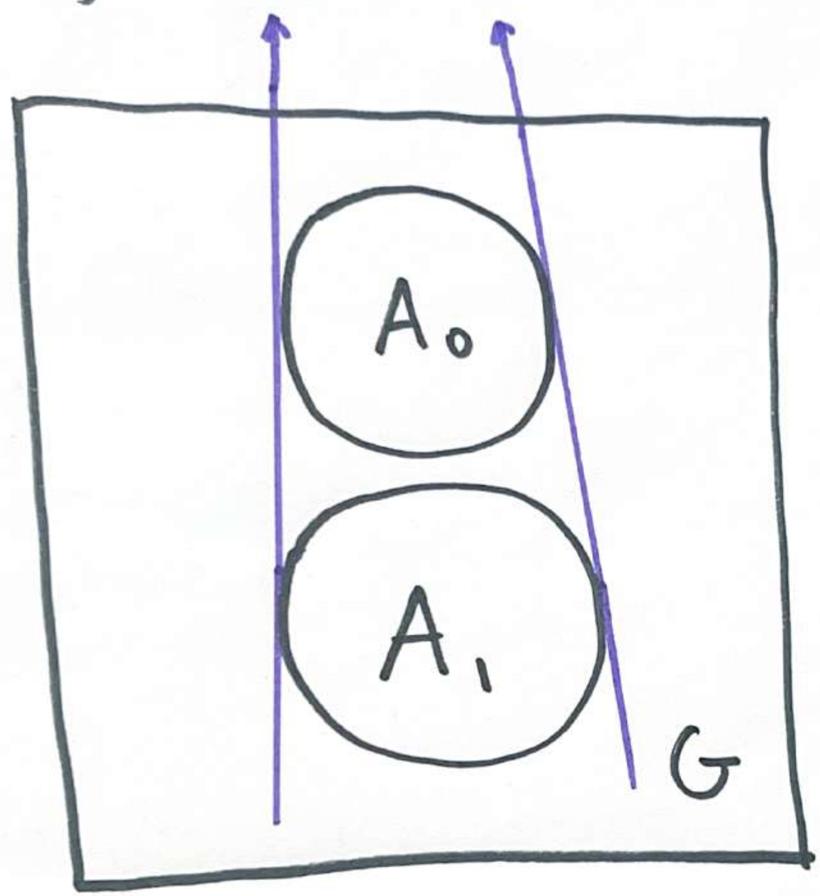

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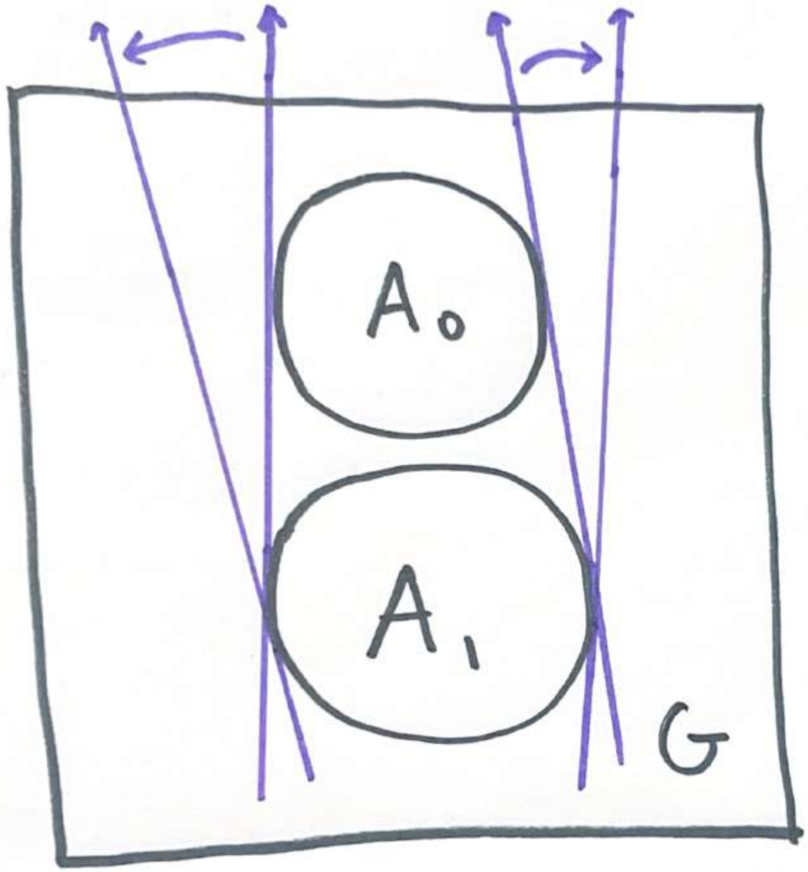

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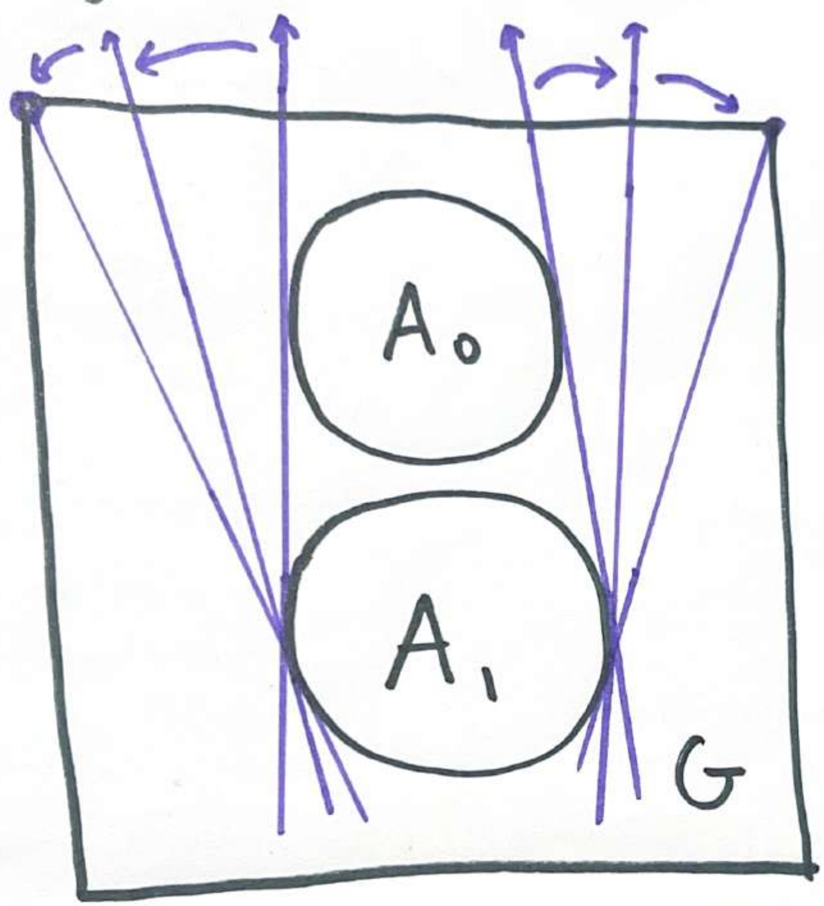

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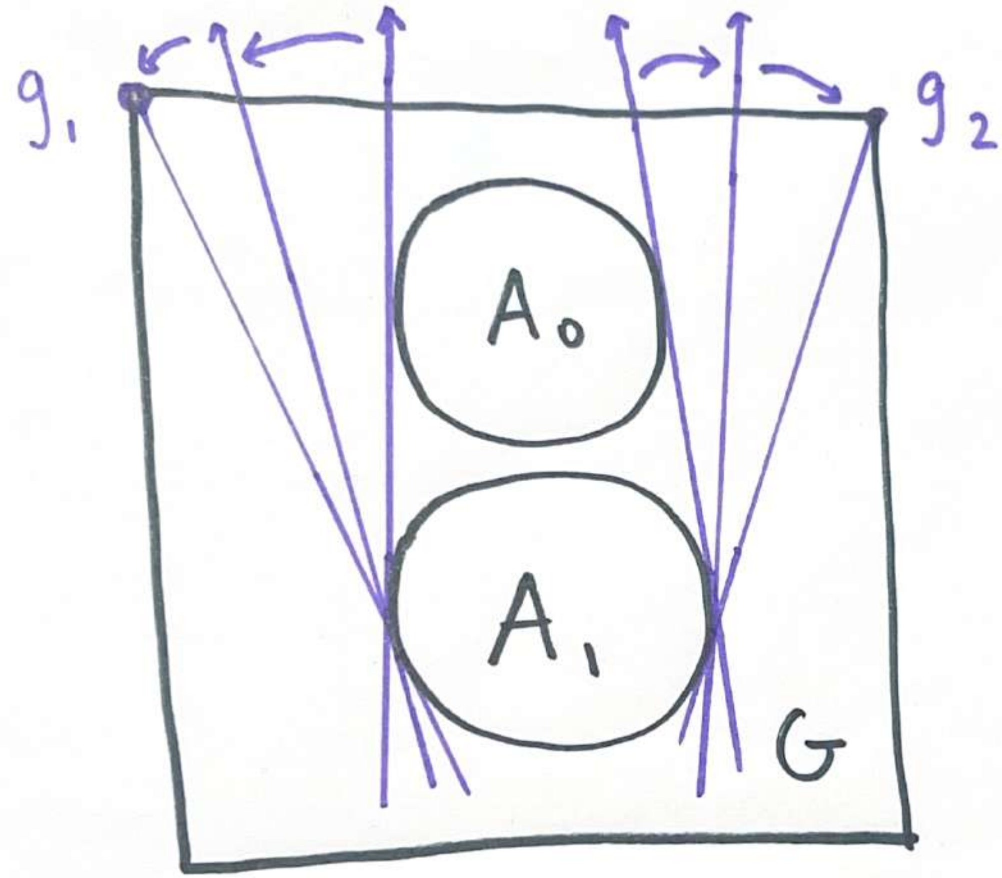
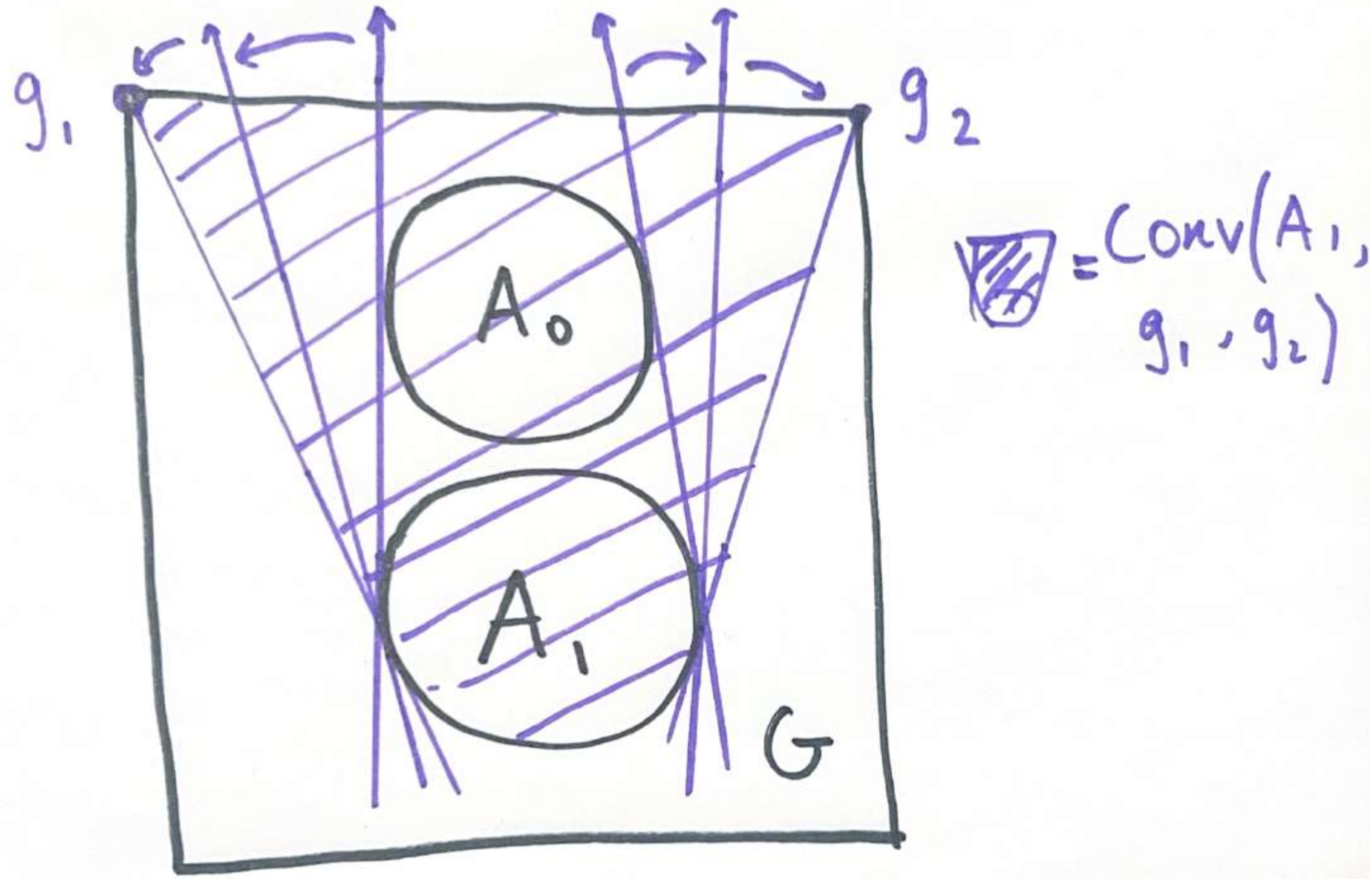


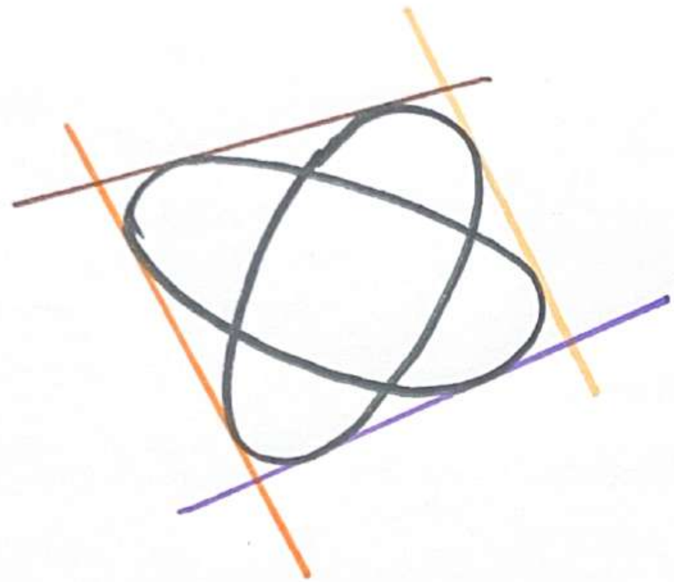
If endpoints of supporting lines intersect vertices of G, we are in luck!

Slide-turning



If endpoints of supporting lines intersect vertices of G, we are in luck!

With multiple supporting lines, each one can slide until the next:



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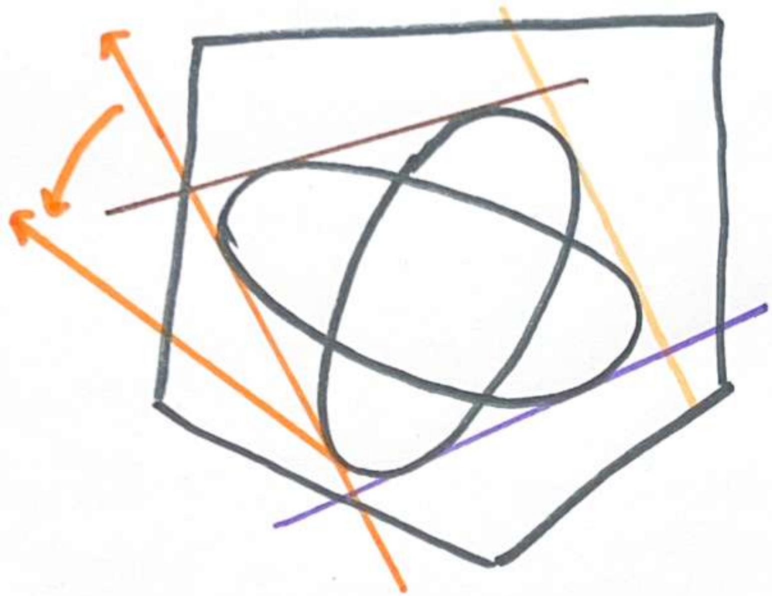

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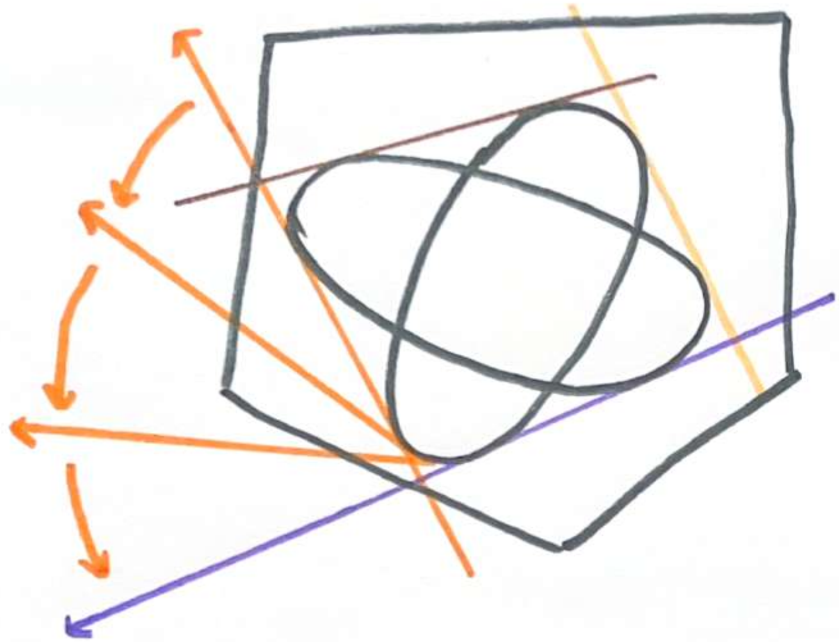

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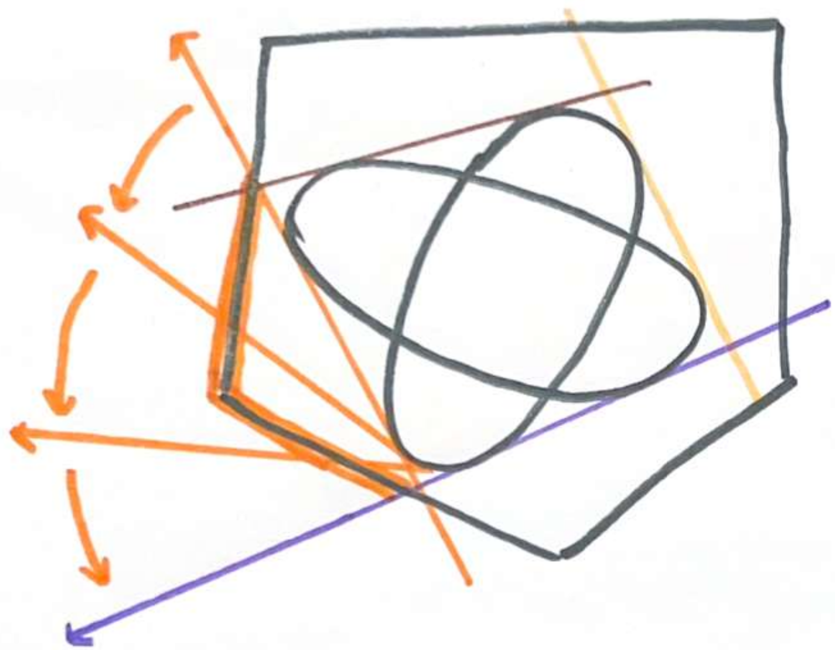

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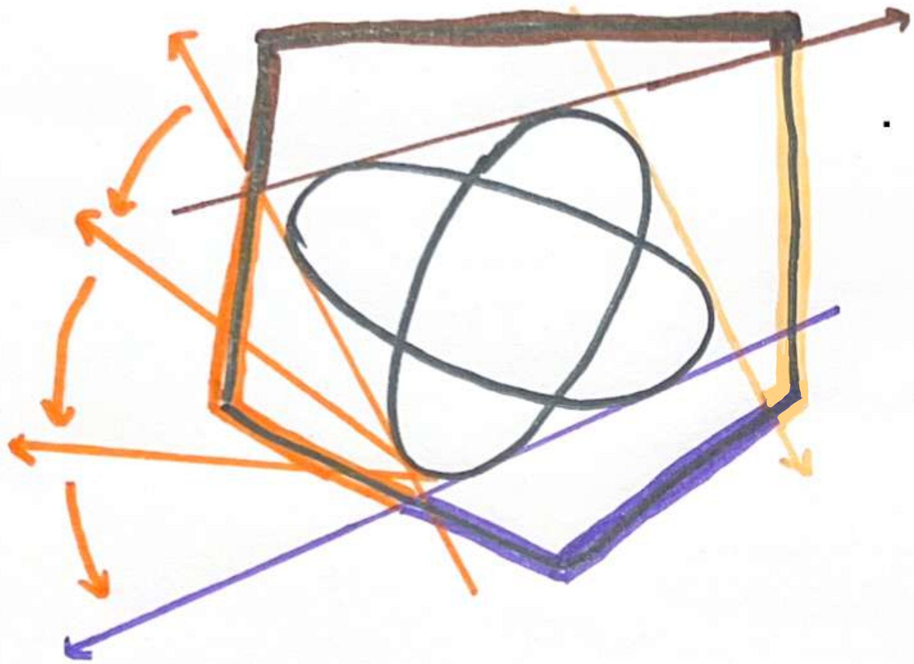

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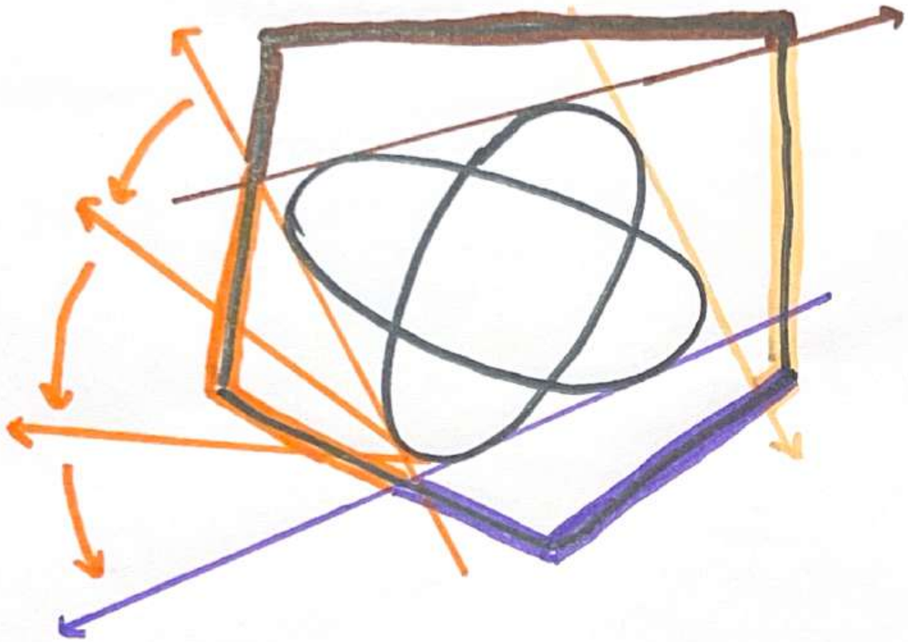
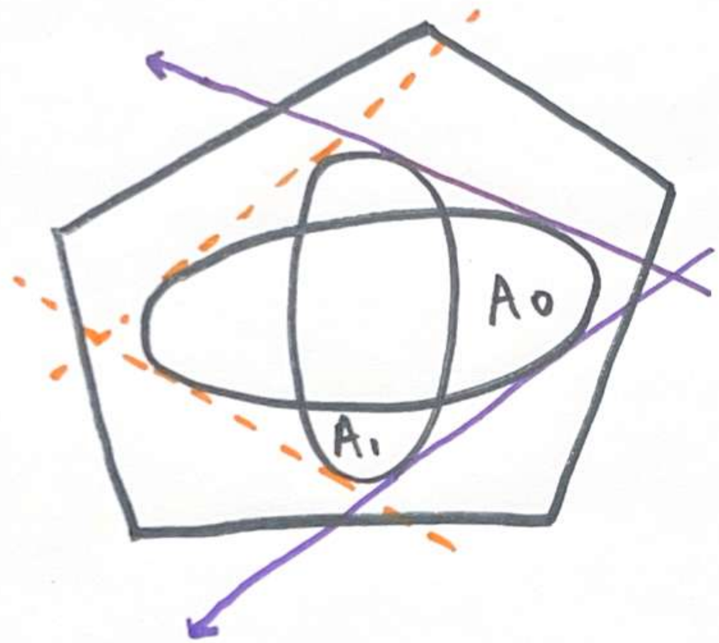


For a fixed orientation, the supporting lines sweep the full boundary of G, hence all vertices.

New goal : Find an <u>adjacent</u> pair of supporting lines that each sweep to a vertex of G.



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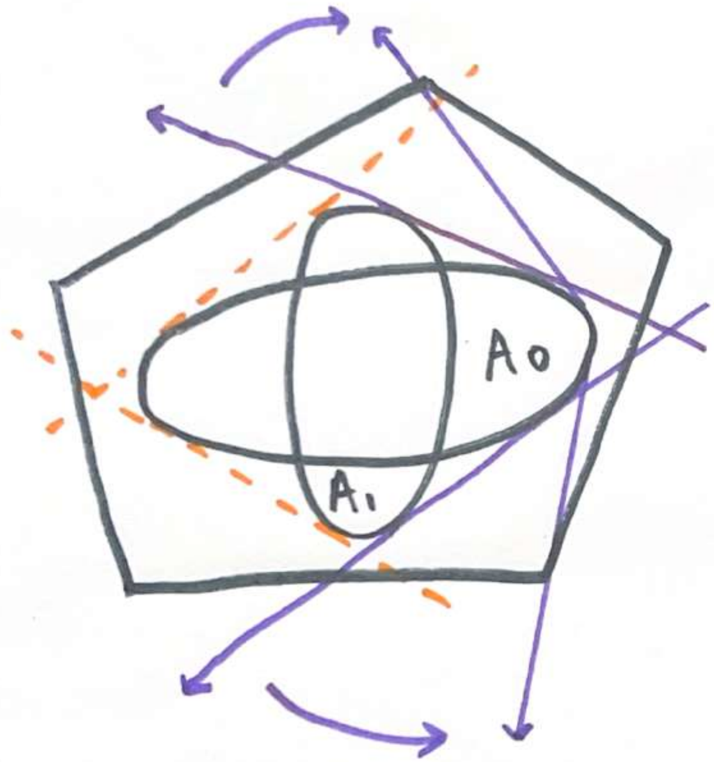

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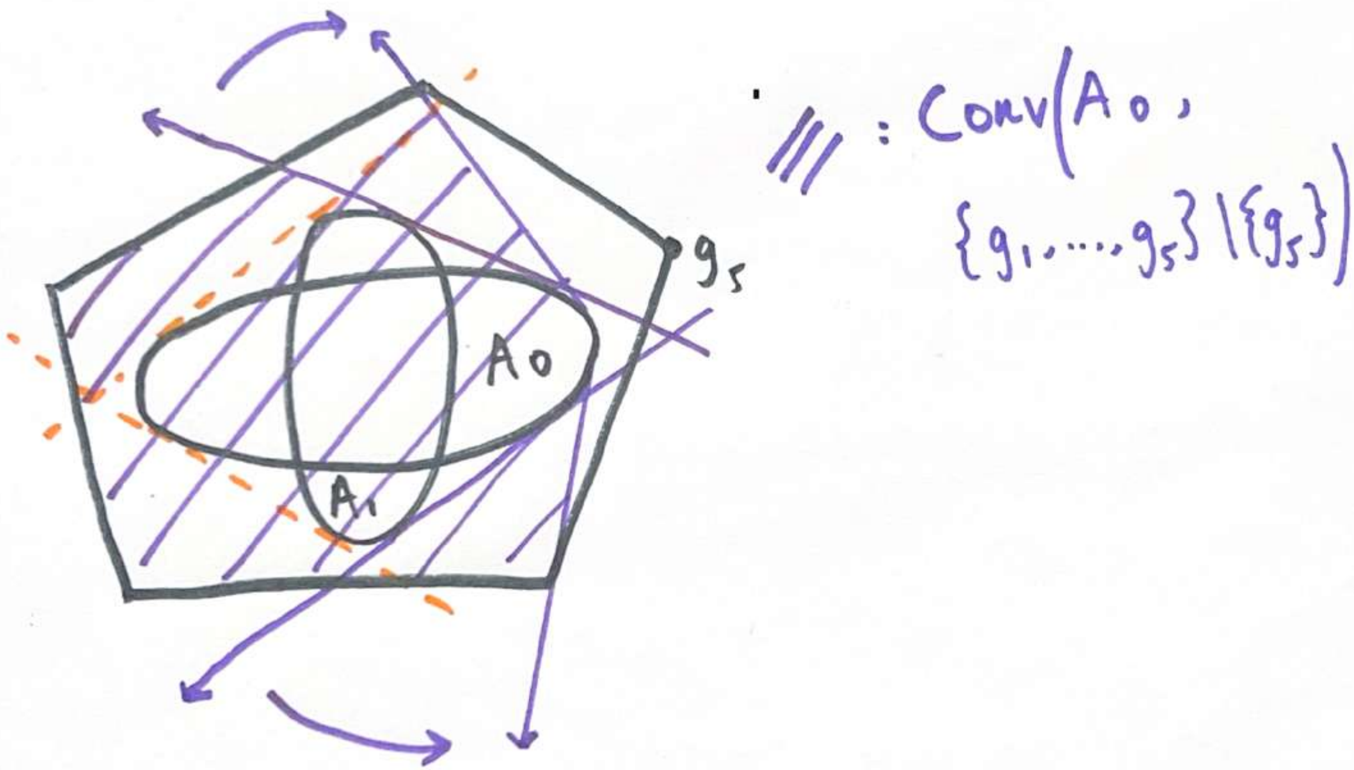

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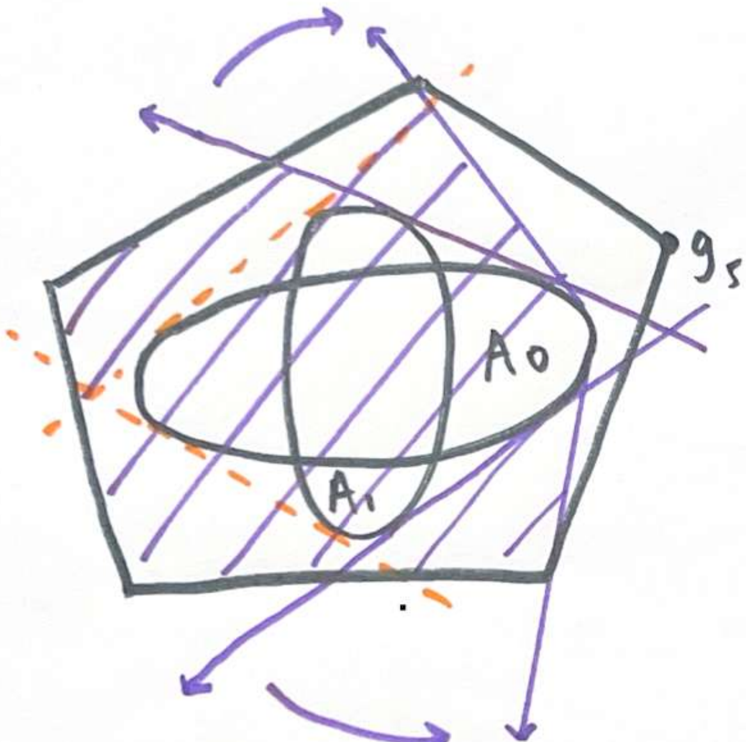

/// : $\mathrm{Conv}(A_0, \{g_1, \dots, g_5\} \setminus \{g_5\})$

If $\#\left\{\text{common supporting lines}\right\} < \#\left\{\text{vertices of } G\right\}$, these always exist.

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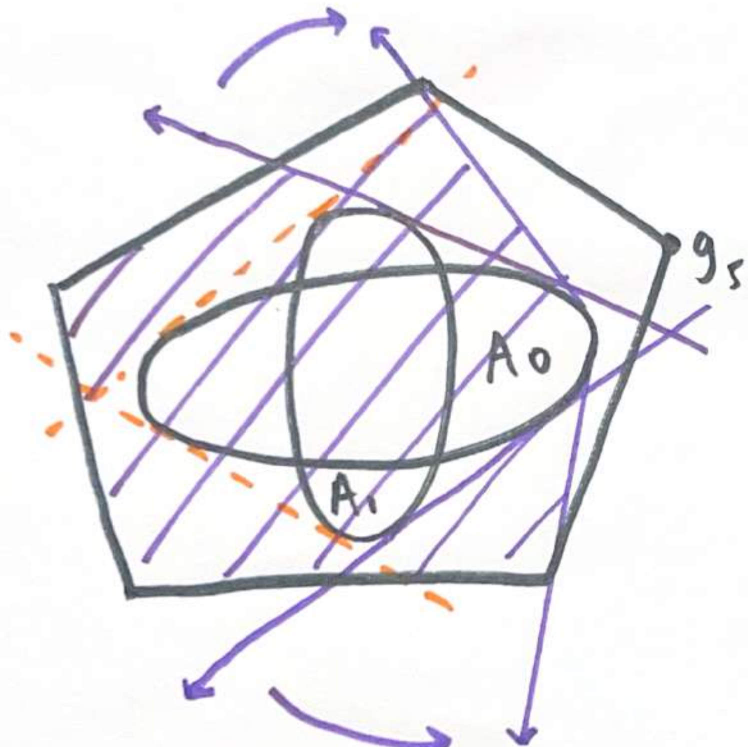

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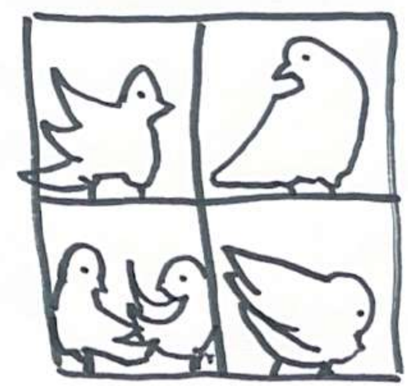

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Cor. $\mathcal{A} = \{A_0, A_1\}$ compact subsets of $\mathbb{R}^2$.
If ∂A_0 and ∂A_1 are smooth plane curves of degree d_1, d_2, and G is a convex n-gon with

$$n > d_1(d_1-1)(d_2-1)d_2,$$

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

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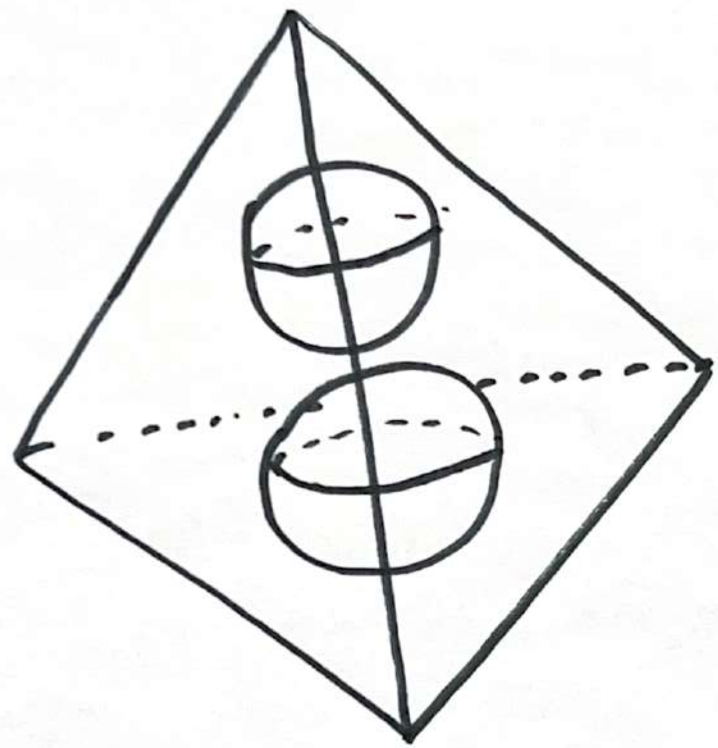
Cor. (Two disks, triangle)

(Two ellipses, pentagon)

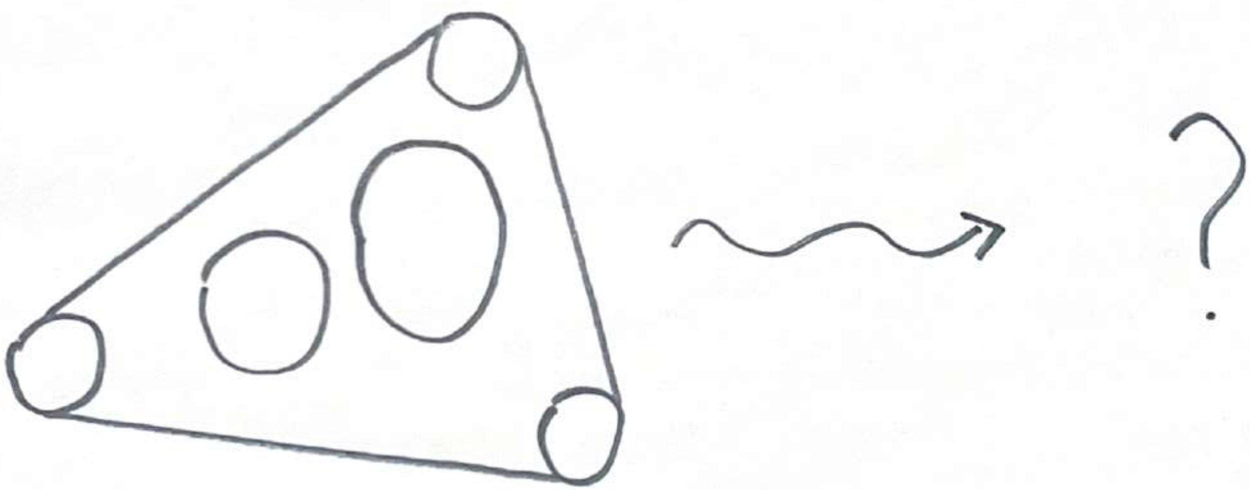
Generalizations fail for $\mathbb{R}^{>2}$.

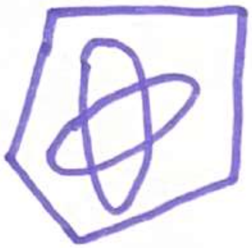


(Czédli., 2017).

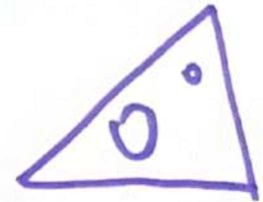
Returning to convex geometries.

Instead of $G = \mathrm{Conv}(\text{points})$, want a theorem for $G = \mathrm{Conv}\left(\begin{array}{c}\text{other convex}\\ \text{compact shapes}\end{array}\right)$





Thank you!



arXiv: 2512.14972.

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