MCDUFF–SEGAL'S ARGUMENT FOR GROUP COMPLETION

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Abstract. We present the details in the proof of the group completion theorem by McDuff–Segal [MS76].

Contents

1. Introduction

Given a topological monoid M, recall that we can construct its classifying space BM by regarding M as a one-object category and taking the geometric realization of its nerve. When M is grouplike, i.e. $\pi := \pi_0(M)$ is a group, the functor B can be regarded as some sort of "delooping" of M. That is, the map $M \to \Omega BM$ sending m to the loop corresponding to the 1-simplex of BM labeled by m is a homotopy equivalence. Consequently, the induced map $H_*(M) \to H_*(\Omega BM)$ is an isomorphism. However, if M is just a monoid, then $H_*(M)$ and $H_*(\Omega BM)$ are not isomorphic via this map.

It turns out that the only obstruction to the isomorphism is that $\pi := \pi_0(M)$ is not a group. But it remains a multiplicatively closed subset, thanks to the monoid operation. So perhaps we should localize at its image in $H_0(M)$. This is exactly what we missed:

Theorem 1.1 (Group completion)**.** *If π is in the center of $H_*(M)$, then*

$$H_*(M)[\pi^{-1}] \to H_*(\Omega BM)$$

is an isomorphism.

This theorem has many applications across geometric topology and K-theory. In this note, we will expound on the nice proof given in the paper [MS76].

2. Various constructions and definitions

Before moving on to the proof, we need to define a few essential pieces of machinery.

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2.1. **Homotopy quotient by a monoid.** Let M be a topological monoid acting on a space X. If M was a topological group, then we could have formed the homotopy quotient X_M by taking the universal bundle $EM \to BM$ and setting $X_M := (X \times EM)/M$. Then X_M has a natural map to BM, given by projection onto the second factor, with fiber X. Even when M is not a group, it is still possible to construct a space with these properties. First consider the category $\mathcal{X}$ with objects X and morphisms $M \times X$ where a pair (m, x) is a morphism from x to mx. The nerve of $\mathcal{X}$ is:

$$\begin{aligned} N_0\mathcal{X} &= X \\ N_1\mathcal{X} &= M \times X \\ N_2\mathcal{X} &= M \times M \times X \\ &\dots \end{aligned}$$

where the kth simplex is given by a string of morphisms

$$x \to m_1 x \to m_2 m_1 x \cdots \to m_k m_{k-1} \cdots m_1 x.$$

We then define $X_M := |N_\bullet \mathcal{X}| = \sqcup_{p\geq 0}(\Delta^p \times X \times M^p)/\sim$ to be the geometric realization of this nerve. The projection ignoring the middle X coordinate still gives a map to BM with fiber X, so we have done our job. (It is worth noting that this map is *not* usually a fibration.)

2.2. **Homotopy fibers and homology fibrations.** If we have an arbitrary map of spaces $p : E \to B$, we can turn it into a fibration as follows. Fix a basepoint $b \in B$. Consider the space

$$E_p := \{(e, \gamma) : e \in E, \gamma : I \to B, \gamma(0) = p(e)\}$$

considered as a subspace of $E \times \mathrm{Map}(I, B)$. Then:

Proposition 2.1. *$E_p \to B, (e, \gamma) \mapsto \gamma(1)$ is a fibration.*

Proof. Suppose we have a homotopy $g_t : X \to B$ and a lift $\tilde{g}_0 : X \to E_f$. We can write $\tilde{g}_0(x) = (h(x), \gamma_x) \in E_p$, for some $h : X \to E$ and $\gamma_x : I \to B$. Then we get a lift $\tilde{g}_t(x) := (h(x), \gamma_x * g_{[0,t]}(x))$, where $g_{[0,t]}$ is the homotopy g ran until time t. □

Now fix a basepoint $b \in B$. Then E includes into the space E_p via $e \mapsto (e, c_{p(e)})$ where $c_{p(e)}$ is the constant map at $p(e)$. Moreover, E_p deformation retracts onto E just by retracting the given path γ until it is the constant map at b. Therefore, under this identification $E \simeq E_p$, we have a fibration $E \to B$ whose fiber over a point b we will call the *homotopy fiber of $p : E \to B$*, denoted $\mathrm{hofib}(p)_b$. Like many homotopical constructions, these are only unique up to homotopy equivalence. Let us follow with some useful facts about homotopy fibers, the first of which follows from this discussion.

Lemma 2.2. *The homotopy fiber of a map $p : E \to B$ is the pullback of p along the the pathspace fibration $f : PB \to B$.* □

Lemma 2.3. *If E_p is contractible, then* $\mathrm{hofib}(p)_b \simeq \Omega B_b$.

Proof. Consider first the map $i : \{*\} \hookrightarrow B$ with $i(*) = b$. Following the construction, we have $E_i = \{(*, \gamma) : \gamma : I \to B, \gamma(0) = B\}$ and this is clearly homeomorphic to $PB_b = \{\gamma : I \to B : \gamma(0) = b\}$. The map $E_i \cong PB_b \to B$ is given by evaluating $\gamma \mapsto \gamma(1)$. The fiber of this map over b consists of loops in B based at b, so $\mathrm{hofib}(i)_b \cong \Omega B$.

PB_b is contractible by shrinking each path to the constant path based at b, so we have some homotopy equivalence $PB_b \to E_p\}$, since $E_p \simeq E$ is contractible. We then get a commutative diagram

$$\begin{tikzcd} \mathrm{hofib}(i)_b \cong \Omega B_b \arrow[r, hook] \arrow[d, dashed] & E_i \cong PB_b \arrow[r, "\gamma\mapsto\gamma(1)"] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ \mathrm{hofib}(p)_b \arrow[r, hook] & E_p \arrow[r] & B \end{tikzcd}$$

By taking the long exact sequence on homotopy groups, and applying the five lemma, we see that the dotted arrow is a homotopy equivalence. □

Lemma 2.4. *If $f : E \to B$ is a fibration, then $f^{-1}(b) \simeq$* $\mathrm{hofib}(f)_b$.

Proof. Let $p : E_f \to B$ be the fibration constructed previously. Since both f and p are fibrations, we get two fiber sequences assembling into a commutative diagram:

$$\begin{tikzcd} \mathrm{hofib}(f)_b \arrow[r, hook] \arrow[d, dashed] & E_p \arrow[r, "p"] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ F \arrow[r, hook] & E \arrow[r, "f"] & B \end{tikzcd}$$

and the rest of the argument is the same as in the previous lemma. □

In the cases where p is not as nice as a fibration, we can still define:

Definition 2.5. We say a map $p : E \to B$ is a *homology fibration* if the map $p^{-1}(b) \to \mathrm{hofib}(p)_b$ is a homology equivalence for all basepoints $b \in B$.

By 2.4, all fibrations are homology fibrations.

2.3. **Localization and right fractions.** Now to do some honest algebra. Let R be a ring and let S be a multiplicative subset of R, i.e. contains 1, closed under multiplication. Then we can construct the localization $S^{-1}R$, being mindful of the fact that R need not be commutative, so things are a little weird.

Definition 2.6. $S^{-1}R$ can be constructed *by right fractions* if every element of it can be written as rs^{-1}, and $r_1/s_1 = r_2/s_2$ iff there exists elements t_1, t_2 in S such that $r_1t_1 = r_2t_2$ and $s_1t_1 = s_2t_2$.

This is a somewhat technical condition, slightly weaker than S being contained in the center of R. It is actually necessary to stipulate the first phrase, that the element can be written as rs^{-1}, as R is not necessarily commutative.

Lemma 2.7. *If S is contained in the center of R, then $S^{-1}R$ can be constructed by right fractions.*

Proof. Since S is in the center, we do not need to worry about distinguishing between $s^{-1}r$ and rs^{-1}. First suppose $r_1/s_1 = r_2/s_2$. Then there exists $q \in S$ such that $qr_1s_2 = qr_2s_1$. Let $t_1 := qs_2$ and $t_2 := qs_1$. Then $r_1t_1 = r_1qs_2 = qr_1s_2 = qr_2s_1 = qs_1r_2 = r_2t_2$, and $s_1t_1 = s_1qs_2 = s_2qs_1 = s_2t_2$.

Conversely, suppose such t_1, t_2 exist. Then let $q = t_1t_2$, from which we obtain $qr_1s_2 = r_1t_1s_2t_2 = r_2t_2s_1t_1 = qr_2s_1$ as desired. □

This somewhat contrived definition of "right fractions" is needed for the following proposition:

Lemma 2.8. *If $R[s^{-1}]$ can be constructed by right fractions, then*

$$\mathrm{colim}(R \xrightarrow{\cdot s} R \xrightarrow{\cdot s} \cdots) \cong R[s^{-1}]$$

Proof. We can represent elements of the colimit as pairs (r, k) where r is the element in the kth copy of R. Thus we have the relation $(r, k) \sim (rs, k + 1)$. Consider the map from the colimit to $R[s^{-1}]$ given by $(r, k) \mapsto r/s^k$, which is clearly surjective. If $(r, k) \mapsto 0$, then $r/s^k \sim 0/s^0$, and because $R[s^{-1}]$ can be constructed by right fractions, there exist s^n, s^m such that

$$rs^n = 0s^m = 0, \quad s^ks^n = s^0s^m.$$

From this and the relation we obtain $(r, k) \sim (rs^n, k+n) = (0, m) \sim (0s^m, 0) = (0, 0)$, so the map is injective too. □

3. Proposition 2 implies group completion

Let's see how the proof usually goes if M is a group, in which case we call it G. It follows very quickly from the commutative diagram of the form

$$\begin{tikzcd} G \arrow[r] \arrow[d, dashed] & EG \arrow[r] \arrow[d, "\simeq"] & BG \arrow[d, "\simeq"] \\ \Omega BG \arrow[r] & PBG \arrow[r] & BG \end{tikzcd}$$

where the top row is the principal G-bundle and the bottom row is the path space fibration. We have a homotopy equivalence in the middle arrow because path spaces are contractible. Then everything commutes, and the five lemma finishes us off. The issue when M is not a group is that the top row of this diagram doesn't exist, at least not with homotopy fiber M, so we need to try something else. Let us begin with a proposition.

Proposition 3.1 (Proposition 2. in [MS76])**.** *If X is a space with the action of a topological monoid M, and for all $m \in M$, $x \to m \cdot x$ is homology equivalence, then $X_M \to BM$ is a homology fibration.*

The proof is slightly tricky, so we will save that for later and first see how group completion follows from this. We will first prove the group completion theorem for $\pi_0(M) \cong \mathbf{N}$, then generalize to the case where $\pi_0(M)$ is an arbitrary monoid.

3.1. **The case where $\pi := \pi_0(M)$ is the monoid N.** In nature, the prototypical monoid is $\mathbf{N}$, so it perhaps makes sense to start here. Choose m in the component of the generator $\mathbf{1} \in \mathbf{N}$, and let X in 3.1 be $M_\infty = \operatorname{colim}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots)$. Since homology preserves filtered colimits, we have

$$H_*(X) \cong \operatorname{colim}(H_*(M) \xrightarrow{\cdot m_*} H_*(M) \xrightarrow{\cdot m_*} \cdots) \cong H_*(M)[m_*^{-1}]$$

where the last isomorphism is due to 2.8. Now since we've picked $m \in 1 \in \pi$, the induced map $\cdot m_*$ in homology corresponds to $\mathbf{1} \in \pi = \langle \mathbf{1} \rangle$, hence

$$H_*(X) \cong H_*(M)[m_*^{-1}] \cong H_*(M)[\mathbf{1}^{-1}] \cong H_*(M)[\pi^{-1}]$$

Finally, consider the left action of M on X. For $n \in M$, the action is by multiplication, and this induces the following commutative diagram on homology:

$$\begin{tikzcd} H_*(M) \arrow[r, "\cdot m_*"] \arrow[d, "n_* \cdot"] & H_*(M) \arrow[r, "\cdot m_*"] \arrow[d, "n_* \cdot"] & \cdots \\ H_*(M) \arrow[r, "\cdot m_*"] & H_*(M) \arrow[r, "\cdot m_*"] & \cdots \end{tikzcd}$$

but since $n_* \in \pi$, multiplication by n_* is invertible in $H_*(M_\infty) = H_*(M)[\pi^{-1}]$, so the action of M on X is a homology equivalence. So by 3.1, we conclude that $p : X_M = (M_\infty)_M \to BM$ is a homology fibration, i.e. $p^{-1}(b) \to \operatorname{hofib}(p)_b$ is a homology equivalence.

It remains to figure out what $p^{-1}(b)$ and $\operatorname{hofib}(p)_b$ are. By the previous section, the homotopy quotient X_M is the realization of the nerve $N_*\mathcal{X}$ whose k-simplex is

$$N_k\mathcal{X} = M^k \times M_\infty = M^k \times \operatorname{colim}(M \to M \to \cdots) = \operatorname{colim}(M^{k+1} \to M^{k+1} \to \cdots)$$

where the last equality is again because filtered colimits commute with products. Then, because geometric realization commutes with colimits, we have

$$\begin{aligned} X_M = (M_\infty)_M &= \left| \; N_0\mathcal{X} \Longleftarrow N_1\mathcal{X} \Lleftarrow N_2\mathcal{X} \cdots \right| \\ &= \left| \operatorname{colim}(M \to \cdots) \Longleftarrow \operatorname{colim}(M^2 \to \cdots) \Lleftarrow \cdots \right| \\ &= \left| \operatorname{colim}\left(\; M \Longleftarrow M^2 \Lleftarrow \cdots \right) \right| \\ &= \operatorname{colim} \left| \; M \Longleftarrow M^2 \Lleftarrow \cdots \right| \\ &= \operatorname{colim}(EM) \end{aligned}$$

The filtered colimit of a sequence of contractible spaces is contractible, hence X_M is too. Therefore by 2.3, $p : X_M \to BM$ has homotopy fiber ΩBM. The regular fiber of the map $(M_\infty)_M \cong \operatorname{colim}(EM) \to BM$ is just M_∞, since on the level of nerves, we have projections

$$N_k\mathcal{X} = M^k \times M_\infty \to M^k$$

whose fibers are M_∞, and taking the realization of the constant simplicial space at M_∞ gives M_∞. Looking back to the homology fibration definition, we thus obtain a homology isomorphism

$$H_*(M)[\pi^{-1}] \cong H_*(X) \cong H_*(p^{-1}(b)) \to H_*(\operatorname{hofib}(p)_b) \cong H_*(\Omega BM)$$

as desired.

3.2. **Reducing to π finitely generated.** Suppose instead now that π is generated by some $s_1, \ldots, s_k$. Then setting $s := s_1 s_2 \cdots s_k$, and letting $X = \operatorname{colim}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots)$ for some $m \in s$, by the same argument in the previous section we have

$$H_*(X) \cong H_*(M)[m_*^{-1}] \cong H_*(M)[s^{-1}] \cong H_*(M)[\pi^{-1}]$$

The action of M on X is a homology isomorphism by the same reason as above, and the rest of the argument proceeds identically. Unfortunately, not every monoid is finitely generated. Fortunately,

Theorem 3.2. *Every monoid $(M, \cdot)$ is the quotient of a free monoid by a set of relations. Moreover, M is isomorphic to a filtered colimit of finitely generated monoids.*

Proof. Let A be the generating set of M (it may be that $A = M$). Let FA be the free monoid on A. Let R consist of all equations satisfied by the elements of A. Then we have a map $FA \to M$ given by sending each letter of A to the corresponding element in M, and concatenation of letters to monoid multiplication. This map is clearly surjective, and it factors through R by definition, so $FA/R \cong M$.

Now consider the category P whose objects are pairs (A_0, R_0) where $A_0 \subseteq A$ and $R_0 \subseteq R$ are finite subsets such that all equations in R_0 consist only of elements in A_0. The morphisms are given by inclusion. This category is filtered; it is clearly nonempty, and for any pair of elements $(A_0, R_0), (A_1, R_1) \in P$ includes into $(A_0 \cup A_1, R_0 \cup R_1)$. We then have a functor $\mathcal{F} : P \to \mathsf{Monoid}$ sending (A_0, R_0) to the monoid A_0/R_0, and morphisms in P to the inclusion map in Monoid.

We claim that $M \cong \operatorname{colim}(\mathcal{F})$. For any class $[x] \in \mathcal{F}(A_0, R_0) = A_0/R_0$, there is a map sending $[x]$ to the element $x \in M$. So we have maps from each element of the diagram $\mathcal{F}$ to M, so by the universal property of colimits this gives some map $\Phi : \operatorname{colim}(\mathcal{F}) \to M$. Surjectivity is obvious. For injectivity, suppose $[x] \in \operatorname{colim}(\mathcal{F})$ maps to zero. The class $[x]$ is represented by some $x \in FA_0/R_0$ for some $(A_0, R_0) \in P$, so then $\Phi([x]) = 0$ implies that $x = 0$ is a relation of M. Then the class of x is zero in $(A_0, R_0 \cup \{x = 0\}) \in P$, hence zero in the colimit. $\square$

Now we can assume $\pi = \operatorname{colim}(\mathcal{F})$ for some functor $\mathcal{F}$ mapping into Monoid. Then, for each object (A_i, R_i) in the category P, we have that $\mathcal{F}(A_i, R_i) =: \pi_{(i)}$ is a finitely generated monoid, so we can repeat the same trick in the finitely generated case above to obtain m_i in the component s_i which generates $\pi_{(i)}$. Then we get a space $X_i = \operatorname{colim}(M \xrightarrow{\cdot m_i} \cdots)$ such that

$$H_*(X_i) \cong H_*(M)[(m_i)_*^{-1}] \cong H_*(M)[(s_i)_*^{-1}] \cong H_*(M)[\pi_{(i)}^{-1}]$$

and taking colimits, we have

$$H_*(X) \cong \operatorname{colim}(H_*(X_i)) \cong \operatorname{colim} H_*(M)[\pi_{(i)}^{-1}] \cong H_*(M)[\pi^{-1}]$$

as desired. Then M acts on X by homology equivalences since it acts on each X_i by homology equivalence. And X_M remains contractible since homotopy quotients commute with filtered colimits. This completes the proof of group completion.

4. Proof of Proposition 2

To prove Proposition 2, the authors end up adopting a alternate definition of homology fibration, which they call a *homology-fibration*. This is different because there is a hyphen. But they turn out to be not so different, as the names suggest. In this section we first explain the difference, then complete the proof of Proposition 2.

4.1. **Homology fibration v.s. homology-fibration.**

Definition 4.1. A map $p : E \to B$ is a *homology-fibration* if any $b \in B$ has arbitrarily small contractible neighborhoods U such that $p^{-1}(b') \hookrightarrow p^{-1}(U)$ is a homology equivalence for all $b' \in U$.

Recall that earlier in the proof all we really needed was for the map $X_M \to BM$ to be a homology fibration. We know that BM can be modeled as the geometric realization of a simplicial set, which means that BM can always be given a CW structure. In particular, CW complexes are always paracompact and locally contractible. Under these assumptions, we can forget the hyphen:

Proposition 4.2 (Proposition 5, [MS76])**.** *If B is a paracompact, locally contractible space, and $p : E \to B$ is homology-fibration, then it is a homology fibration.*

To prove this, we will make use of the following proposition, a mix of statements from §4 and §5 in [Seg68].

Proposition 4.3. *Let $\mathscr{B}$ be a basis for a paracompact, locally contractible, and contractible space B. Let $p : E \to B$ be a homology-fibration. Then:*

(1) If C is the category whose elements are finite subsets of $\mathscr{B}$, and morphisms are inclusions, then $|\mathscr{B}| := |NC|$ is homotopy equivalent to B.

(2) *There exists a space $E_{\mathscr{B}}$ homotopy equivalent to E, and a map $E_{\mathscr{B}} \to |\mathscr{B}|$ giving rise to a spectral sequence $E^*_{*,*}$ with*

$$E^2_{p,q} = H_p(|\mathscr{B}|, \mathscr{H}_q) \implies H_{p+q}(E_{\mathscr{B}}) = H_{p+q}(E)$$

where $\mathscr{H}$ is the local coefficient system $U \mapsto p^{-1}(U)$. □

In the remaining discussion, we will abbreviate "basis consisting of contractible sets" by "contractible basis."

Corollary 4.4 (Proposition 6, [MS76])**.** *If $s : X \to Y$ is a homology-fibration, and Y is paracompact, locally contractible, and contractible, then $s^{-1}(y) \to X$ is a homology equivalence for each $y \in Y$.*

Proof. Since s is a homology-fibration, given a contractible basis $\mathscr{Y}$ of Y we know that $s^{-1}(y) \to s^{-1}(U)$ is a homology equivalence for all $U \in \mathscr{Y}$. Observing the $E^2_{0,q}$ term of the spectral sequence from the previous proposition, we obtain a chain of isomorphisms

$$H_q(s^{-1}(y)) \cong H_q(s^{-1}(U)) \cong \mathscr{H}_q(U) \cong H_0(|\mathscr{Y}|; \mathscr{H}_q) \cong H_q(X)$$

where the penultimate isomorphism is because $Y \simeq |\mathscr{Y}|$ is contractible, and the final isomorphism is via the spectral sequence. □

Proof of 4.2. Let $f : PB_b \to B$ be the pathspace fibration. By 2.2, $\mathrm{hofib}(p)_b \simeq f^*E$, we have the pullback square below, where we have also added the fibers on top for $\gamma \in PB_b$ and $b \in B$ satisfying $f(\gamma) = b$:

$$\begin{tikzcd}
(f^*p)^{-1}(\gamma) \arrow[r, "\simeq"] \arrow[d] & p^{-1}(b) \arrow[d] \\
f^*E = \mathrm{hofib}(p)_b \arrow[r] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\
PB_b \arrow[r, "f"'] & B
\end{tikzcd}$$

Importantly, the fibers $(f^*p)^{-1}(\gamma)$ and $p^{-1}(b)$ are equivalent, since we have a pullback.

We claim first that f^*p is a homology-fibration, i.e. for any $\gamma \in PB_b$, there exists a contractible neighborhood $V \ni \gamma$ such that $(f^*p)^{-1}(\gamma) \to (f^*p)^{-1}(V)$ is a homology equivalence. To see this, choose a contractible basis $\mathscr{B}$ of B. Since p is a homology-fibration, the map $p^{-1}(b) \to p^{-1}(U)$ is a homology equivalence for all $U \in \mathscr{B}$. By some point-set technicalities, we can find a contractible basis $\mathscr{B}^*$ of PB_b such that for all $V \in \mathscr{B}^*$, $f(V) \in \mathscr{B}$ and $V \to f(V)$ is a fibration. Now we can restrict the pullback square above to obtain

$$\begin{tikzcd}
f^*p^{-1}(f(V)) \arrow[r, "\pi"] \arrow[d, "f^*p"'] & p^{-1}(f(V)) \arrow[d, "p"] \\
V \arrow[r, "f"] & f(V)
\end{tikzcd}$$

where the bottom arrow is a fibration and homotopy equivalence (since both spaces are contractible). Then this makes the top arrow π a fibration since pullbacks of fibrations are also fibrations. But π is the projection $(v, e) \mapsto e$. The fiber over a point $e \in p^{-1}(f(V))$ with $p(e) = b$ is

$$\begin{aligned}
\pi^{-1}(e) &= \{(v, e) \in f^*p^{-1}(f(V)) : \pi(v, e) = e\} \\
&= \{(v, e) \in V \times p^{-1}(f(V)) : f(v) = p(e) = b\} \\
&\cong \{v \in V \mid f(v) = b\} \\
&= f^{-1}(b)
\end{aligned}$$

Since f is a homotopy equivalence and a fibration, we have by the long exact sequence in homotopy groups that the fibers $f^{-1}(b) = \pi^{-1}(e)$ are contractible. Hence π is a weak homotopy equivalence, hence a homology equivalence. Restricting once again to the fibers:

$$\begin{tikzcd}
(f^*p)^{-1}(\gamma) \arrow[r, "\pi"] \arrow[d, "f^*p"'] & p^{-1}(b) \arrow[d, "p"] \\
V \arrow[r, "f"] & f(V)
\end{tikzcd}$$

we see that the top, right, and bottom arrows are all homology equivalences, and thus so is the left arrow. Hence f^*p is a homology-fibration.

Then, applying 4.4, taking $s = f^*p$, $X = f^*E$, and $Y = PB_b$, we have that $(f^*p)^{-1}(\gamma) \to f^*E$ is a homology equivalence. But by earlier, we have that $(f^*p)^{-1} \simeq p^{-1}(p)$, so actually $p^{-1}(p) \to f^*E = \mathrm{hofib}(p)_b$ is a homology equivalence for all $b \in B$. Hence p is a homology equivalence (without hyphen). □

4.2. **Cleaning up and last steps.** Now back to the proof of 3.1, whose conclusion is that a certain map is a homology fibration. We now know that homology-fibrations are homology fibrations under certain circumstances, so it suffices to prove the conclusion that the map is a homology-fibration.

Definition 4.5. Let X, Y, Z be topological spaces. The *double mapping cylinder* of maps $f : X \to Y$, $g : X \to Z$ is the quotient

$$\mathrm{Cyl}(Z \leftarrow X \to Y) := (X \times [0,1] \sqcup Y \sqcup Z)/(x,0) \sim f(x), (x,1) \sim g(x)$$

When one of Y or Z is the one point space, then we recover the mapping cone. When one of the maps is the identity, we recover the mapping cylinder of the other map. Notice that this construction is functorial: levelwise maps from $Z_1 \leftarrow X_1 \to Y_1$ to $Z_2 \leftarrow X_2 \to Y_2$ induce a map of double mapping cylinders. The advantage of the modified definition, the authors write, is that it "makes the following proposition obvious." Let's see:

Proposition 4.6 (Proposition 3, [MS76]). *Suppose we have a commutative diagram*

$$\begin{tikzcd} E_1 \arrow[d, "p_1"] & E_0 \arrow[l] \arrow[r] \arrow[d, "p_0"] & E_2 \arrow[d, "p_2"] \\ B_1 & B_0 \arrow[l, "f_1"'] \arrow[r, "f_2"] & B_2 \end{tikzcd}$$

where the vertical arrows are homology-fibrations, and $p_0^{-1}(b) \to p_i^{-1}(f_i(b))$ is a homology-equivalence for all $b \in B_0$, then the induced map

$$p : \mathrm{Cyl}(E_1 \leftarrow E_0 \to E_2) \to \mathrm{Cyl}(B_1 \leftarrow B_0 \to B_2)$$

is a homology-fibration.

Proof. Let $c \in \mathrm{Cyl}(B_1 \leftarrow B_0 \to B_2)$. Then there exist arbitrarily small neighborhoods U of c that are equal to mapping cylinders of the form $\mathrm{Cyl}(U_0 \to U_i)$ for $i = 0, 1, 2$, where $U_i \subset B_i$ is open, and this will imply the proposition. To see this, choose a point $x \in \mathrm{Cyl}(B_1 \leftarrow B_0 \to B_2)$. We have a few cases:

(1) x lies entirely within the component of $B_i, i = 1, 2$ of the cylinder. Then we can find an contractible open set $U_i \subset B_i$. Now let $U = f_i^{-1}(U_i)$. Then $x \in \mathrm{Cyl}(f : U \to U_i)$. Then $p^{-1}(x) \to p^{-1}(\mathrm{Cyl}(f : U \to U_i))$ is a homology equivalence since $\mathrm{Cyl}(f : U \to U_i) \simeq U_i$ and this is just the map $p_i^{-1}(x) \to p_i^{-1}(U)$, which is a homology equivalence by assumption.

(2) $x = (b, t)$ lies in the component $B_0 \times (0, 1)$. Then we can find a neighborhood $U \ni b$ in B_0, and some small $\varepsilon > 0$ such that $x \in U \times (t-\varepsilon, t+\varepsilon)$. This is homotopy equivalent to the identity mapping cylinder $\mathrm{Cyl}(U \to U)$. Just like before, the map $p^{-1}(x) \to p^{-1}(\mathrm{Cyl}(U \to U))$ is the map $p_0^{-1}(x) \to p_0^{-1}(\mathrm{Cyl}(U \to U)) \simeq p_0^{-1}(U)$, which is a homology equivalence by assumption.

(3) x lies in the gluing points, i.e. $x = (b, 0) \sim f_1(b)$ or $x = (b, 1) \sim f_2(b)$ for $b \in B_0$. In the former case, we can take some open $U_1 \ni f_1(b)$ in B_1, and let $U = f_1^{-1}(U_1)$. Then consider $U_1 \cup_{f_1} U \times [0, \varepsilon) \simeq \mathrm{Cyl}(U \to U_1)$. Then $p^{-1}(x) \to p^{-1}(\mathrm{Cyl}(U \to U_1)) \simeq$ is the map

$$(p_0^{-1}(b), 0) \xrightarrow{\text{homology eq.}} p_1^{-1}(f_1(b)) \to p_1^{-1}(U_1)$$

which is a homology equivalence.

This completes the proof. □

In the next proposition, $||-||$ denotes the *fat* geometric realization functor, which is the same as the regular realization except that we do not quotient out by the degeneracy maps. The fat realization of a simplicial space E is homotopy equivalent to the usual realization, as long as the space is *good*, that is if $E_k \hookrightarrow E_{k+1}$ is a closed cofibration, which is the case for all CW complexes.

We will use $||E||^{(n)}$ to denote the image of $\sqcup_{k \le n} \Delta^k \times E_k$ in $||E||$. We will call $||E||^{(n)}$ the "n-skeleton" of $||E||$. This allows to induct on n to make arguments about $||E||$, which is defined as the colimit of $(||E||^{(0)} \to ||E||^{(1)} \to \cdots)$, as we will see below. The reference for this material is §1 and the appendix in [Seg74].

Proposition 4.7 (Proposition 4, [MS76]). *Suppose $p : E \to B$ is a map of simplicial spaces such that $E_k \to B_k$ is a homology-fibration for all $k \geq 0$. Suppose also that for any simplicial map $\theta : [k] \to [l]$, and any $b \in B_l$, the map $p_l^{-1}(b) \to p_k^{-1}(\theta^* b)$ is a homology equivalence. Then the induced map $p_* : ||E|| \to ||B||$ is a homology-fibration.*

Proof. We will prove this by inducting on the skeletons. For $k = 0$, the homology-fibration $E_0 \to B_0$ directly yields the homology-fibration $||E||^{(0)} \to ||B||^{(0)}$.

Now suppose that the restricted induced map $p_* : ||E||^{(n-1)} \to ||B||^{(n-1)}$ is a homology-fibration. The n-skeleton of $||E||$ is given by the pushout

$$\begin{tikzcd}
\partial\Delta^k \times E_k \arrow[r, hook] \arrow[d] & \Delta^k \times E_k \arrow[d] \\
||E||^{(k-1)} \arrow[r, hook] & ||E||^{(k)}
\end{tikzcd}$$

but this is just the double mapping cylinder

$$\mathrm{Cyl}(||E||^{(k-1)} \leftarrow \partial\Delta^k \times E_k \to \Delta^k \times E_k)$$

and likewise for B, so we can assemble these maps into a commutative diagram

$$\begin{tikzcd}
||E||^{(k-1)} \arrow[d, "p_*"] & \partial\Delta^k \times E_k \arrow[l] \arrow[r, hook, "i_E"] \arrow[d, "\mathrm{id} \times p_k"] & \Delta^k \times E_k \arrow[d, "\mathrm{id} \times p_k"] \\
||B||^{(k-1)} & \partial\Delta^k \times B_k \arrow[l] \arrow[r, hook, "i_B"] & \Delta^k \times B_k
\end{tikzcd}$$

satisfying the assumptions of the previous proposition:

(1) The vertical arrows are homology-fibrations. This is by the inductive hypothesis and the assumption that $E_k \to B_k$ is a homology-fibration for all k.
(2) For $x \in \partial\Delta^k \times B_k$, the top right arrow is a homeomorphism $(\mathrm{id} \times p_k)^{-1}(x) \xrightarrow{i_E} (\mathrm{id} \times p_k)^{-1}(i_B(x))$, hence a homology equivalence.
(3) The top left arrow is a homology equivalence by the inductive hypothesis.

By the previous proposition, we therefore have a homology-fibration $||E||^{(k)} \to ||B||^{(k)}$ for all $k \geq 0$, so $||E|| \to ||B||$ is a homology-fibration. □

Finally, we come to the proof of Proposition 2 (3.1):

Proof. We will first use 4.7 with $||E|| \simeq |E| = X_M$, and $||B|| \simeq |B| = BM$. Recall that in this case E and B are the simplicial sets with $E_k = X \times M^k$ and $B_k = M^k$. The map $E_k \to B_k$ is just projection, hence a homology-fibration. For the second assumption, it suffices to check for θ a coface/codegeneracy map. So let $\theta : [k] \to [k+1]$ be a coface map. Then we have the diagram

$$\begin{tikzcd}
{[k+1]} \arrow[r] & X \times M^{k+1} \arrow[r, "p_{k+1}"] \arrow[d, "\theta^*"] & M^{k+1} \arrow[d, "\theta^*"] & \ni m = (m_1, \dots, m_{k+1}) \arrow[d, mapsto] \\
{[k]} \arrow[u, "\theta"] \arrow[r] & X \times M^k \arrow[r, "p_k"'] & M^k & \theta^*(m) = (m_1, \dots, m_j m_{j+1}, \dots, m_{k+1})
\end{tikzcd}$$

but now $\theta^* : p_{k+1}^{-1}(m) \to p_k^{-1}(\theta^*(m))$ is a map $X \times \{m\} \to X \times \{\theta^*(m)\}$ induced by the monoid multiplication, hence a homology equivalence by the assumptions of Proposition 2. If θ is a codegeneracy map the proof proceeds identically. Hence 4.7 implies that $||E|| = X_M \to ||B|| = BM$ is a homology-fibration. By 4.2, since BM is a CW complex, it is paracompact and locally contractible, so $X_M \to BM$ is a homology fibration, no hyphen. □

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