THE COBAR CONSTRUCTION AND FUNDAMENTAL GROUP

A thesis submitted by
Yiming Song

to the Department of Mathematics at
Columbia University
on March 31, 2026

in partial fulfillment of the honors requirements
for the degree of Bachelor of Arts in Mathematics

Contents

Acknowledgements

- I would first like to thank my advisor, Søren Galatius. Thank you for being so generous with your time. This thesis would not have been written without your sage guidance and insight.
- Thanks to the mathematics department here. In particular, thanks to Michael Thaddeus for teaching and supervising great courses, Azélie Picot for teaching me about simplicial sets, and the Undergraduate Math Society for lots of free food.
- I thank my friends for making these years wonderful.
- Lastly, and mostly, I thank my family for their uncompromising love and support.

1. Introduction

It would be hung in the attics, she thought; it would be destroyed. But what did that matter? she asked herself, taking up her brush again.

—Virginia Woolf, *To the Lighthouse*

Associated to any based topological space (X, x) is its fundamental group $\pi_1(X, x)$. In the seminal 1895 paper *Analysis Situs* [Poi95], Henri Poincaré introduced the fundamental group alongside other important concepts such as homology and cobordism, essentially founding the modern study of algebraic topology. This thesis is broadly concerned with the relationship between the fundamental group and homology: the former contains information that the latter cannot see. One can ask whether it is possible to learn about the fundamental group given information about homology.

While the integral homology groups are always abelian, the fundamental group is often not. So let us instead examine the integral group ring $\mathbf{Z}\pi_1(X, x)$. This is unfortunately an unwieldy object: if $\pi_1(X, x)$ is infinite, which is often the case, then $\mathbf{Z}\pi_1(X, x)$ is of infinite rank. To turn it into something feasible to study, let us consider the quotients of $\mathbf{Z}\pi_1(X, x)$ by the nth powers of its augmentation ideal $\mathcal{I}$. A result of Stallings (Theorem 5.1 here, Theorem 5.4 in [Sta75]) relates this truncated fundamental group ring to the E^∞-page of a certain spectral sequence associated to the *cobar construction* on the singular chains of X. The cobar construction is a functor sending differential graded (dg) conilpotent coalgebras to dg algebras. While it can be defined purely algebraically, it has a nice topological interpretation when performed on the singular chains of a topological space. It is through this that we will relate $\mathbf{Z}\pi_1(X, x)/\mathcal{I}^n$ to the homology of X.

1.1. **Main theorem.** Let X be a path-connected topological space and choose basepoints $a, b \in X$. Let $\pi_X(a, b)$ be the set of homotopy classes of paths in X from a to b. In particular when $a = b$ we obtain the fundamental group of X. Notice that $\pi_1(X, a)$ acts on $\pi_X(a, b)$ by path concatenation, making the free abelian group $\mathbf{Z}\pi_X(a, b)$ into a right $\mathbf{Z}\pi_1(X, a)$-module. Consider the following subspace:

$$X(n)^a_b = \{(x_1, \ldots, x_n) \in X^n \mid x_1 = a \text{ or } x_n = b \text{ or } x_i = x_{i+1} \text{ for some } i\} \subset X^n.$$

Our main theorem is the following.

Theorem 1.1. *There is a surjective group homomorphism*

$$\mathbf{Z}\pi_X(a, b)/\mathbf{Z}\pi_X(a, b)\mathcal{I}_a^{n+1} \to H_n(X^n, X(n)^a_b; \mathbf{Z})$$

whose kernel is isomorphic to $\mathbf{Z}$ *if* $a = b$ *and zero otherwise.*

A dual cohomological version of this result with field coefficients is proved by Deligne and Goncharov in [DG05], Proposition 3.4, where they attribute it to Beilinson. A version for X a real manifold and de Rham cohomology is developed by Chen in [Che73]. The version stated here comes from a paper of Looijenga [Loo25], Theorem 1.1, where the proof is quite concise and elegant, but (perhaps to a student) somewhat opaque. Returning to the paper of Deligne and Goncharov, in the discussion following the statement and proof of this result, the authors suggest another possible proof in terms of cosimplicial spaces. We find this avenue more enlightening. In particular, it offers an explanation for why the subspace $X(n)^a_b$ appears.

The goal of this thesis will be to develop the necessary background to pursue this alternate proof, while hopefully clearly explicating the details of each step. Of course, the authors listed above are interested in this theorem for deeper reasons beyond just its statement and proof. We do not claim to understand why, but give a few remarks on this matter at the very end.

1.2. **Organization.** In §2 we give a crash course on the theory of coalgebras. In §3 we develop the basic theory of (co)simplicial objects, in particular cosimplicial topological spaces. We also collect some helpful technical lemmas about (co)simplicial objects. The reader should feel free to skip these first two sections if they are familiar with the theory. Both coalgebras and (co)simplicial objects are crucial in the following §4, where we introduce the cobar construction and Adams' theorem, and define a generalized two-sided version of the cobar construction. In §5 we generalize Stallings' theorem to this two-sided version, and finally use all these ingredients to prove our main theorem in §6.

I have tried to make the thesis readable to anyone who has had slightly more than a course or two in algebraic topology, and working knowledge of (homological) algebra. Here are some prerequisites:

- Algebraic topology at the level of the first two chapters of Hatcher's book [Hat00].
- Homological algebra, in particular the spectral sequences associated to a bicomplex. They unfortunately play a key computational role throughout the thesis. The proof of the main theorem comes from the close examination of one particular spectral sequence. We have included proofs of some auxillary results about these spectral sequences, but the reader could possibly regard them as a black box.
- Basic notions in category theory: functors, natural transformations, adjunctions, (co)limits.
- Some prior exposure to simplicial sets, e.g. the singular-geometric realization adjunction. This will be very helpful for intuition with cosimplicial objects even though we technically develop the theory of cosimplicial objects from scratch. Simplicial sets will also feature heavily in §5, though we really only care when the simplicial set in question is $\mathrm{Sing}(X)$ for a topological space X.

A few last words on convention. (1) Often we are working with a bicomplex E with a totalization $\mathrm{Tot}\,E$. We will often denote the homology of the totalization by just $H_\bullet E$ rather than $H_\bullet \mathrm{Tot}\,E$. (2) The letter R is reserved for commutative unital rings. Other rings appearing may not be commutative nor unital. (3) Normalized or singular chains of a topological space will have coefficients in $\mathbf{Z}$. (4) If Y is a simplicial set then $|Y|$ denotes its geometric realization. (5) "Space" means topological space. (6) All spectral sequences will be second quadrant, and depending on the filtration, the rth differential will have bidegree $(r, 1-r)$ or $(1-r, r)$.

2. Coalgebra crash course

As the name suggests, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the fact that coalgebraic multiplication has its direction reversed can be somewhat mystifying on first sight. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions. We will not attempt to give detailed proofs in this section, for there is much exposition of this material in the literature. We have primarily drawn from the first chapter of [LV12].

Definition 2.1. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct*, such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$

One way to think about the coproduct is that it tells one how to decompose a given element in the module. Consider the following example:

Example 2.2 ([JR79], §2)**.** Let $C = R[x]$, the ring of polynomials over R. Define a coproduct on a basis element x^n via

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

This looks awfully similar to the binomial formula. Indeed, this coproduct tells us all the ways one can decompose a set of size n into two subsets.

The author learned of this example from this MathOverflow comment of Qiaochu Yuan [Yua]. Though it is combinatorial in nature, the way of thinking about coproducts as some sort of decomposition operator will come up again, when we discuss how the subdivision of topological simplices induces a coproduct on singular chains.

Given an associative algebra A, it is intuitive how to compose the product to obtain a map $A^{\otimes n} \to A$. With a coassociative coalgebra (C, Δ) there is an analogous notion. Define the *iterated coproduct* $\Delta^n : C \to C^{\otimes n+1}$ inductively with $\Delta^0 = \mathrm{id}$, $\Delta^1 = \Delta$, and

$$\Delta^n = \underbrace{\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n \text{ operations}} \circ \Delta^{n-1}.$$

Coassociativity says that we could have inserted the coproduct anywhere within the above tensor product of operations. A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Just as a unital associative R-algebra A is one admitting a unital morphism $R \to A$, we say dually that a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R \end{tikzcd}.$$

The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $\eta : R \to C$ a *coaugmentation*, and if it exists, say that C is *coaugmented*. Because η is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ \eta_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$ and call it the *reduced coalgebra*. We can think of $\overline{C}$ as either a submodule or a quotient of C. The reduced coalgebra also has a *reduced coproduct* $\overline{\Delta}$ given by

$$\overline{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1.$$

2.1. **Conilpotency, cofreeness, and the tensor coalgebra.** Let (C, Δ) be a coaugmented coalgebra. Define the *coradical* (sometimes also called *canonical*) filtration on C by:

$$F_0 C = R, \quad F_r C = R \oplus \{x \in \overline{C} : \overline{\Delta}^n(x) = 0 \text{ for } n \geq r\} \text{ for } r \geq 1.$$

Then we say C is *conilpotent* if this filtration is exhaustive. Conilpotency is fundamentally a finiteness condition; if we are continuing with the coproduct-as-decomposition analogy, it says that any element can only be decomposed finitely many times. It will turn out quite important in all which follows.

Definition 2.3. The *cofree* coassociative R-coalgebra over an R-module M is a conilpotent coassociative coalgebra $\mathcal{F}^c M$ equipped with an R-linear map $s : \mathcal{F}^c M \to M$ sending 1 to 0 and satisfying the following universal property:

> *Let B be a conilpotent coalgebra. Then any R-linear map $f : B \to M$ factors through $\mathcal{F}^c M$, i.e. there exists a unique coalgebra morphism $\tilde{f} : B \to \mathcal{F}^c M$ such that $s \circ \tilde{f} = f$.*

As with other objects defined via universal properties, the cofree coalgebra is unique up to unique isomorphism. In categorical language, this functor

$$\mathcal{F}^c : \mathsf{Mod}_R \to \mathsf{conilCoalg}_R$$

is right adjoint to the forgetful functor sending a conilpotent coalgebra to its underlying module. The reason we restrict ourselves to conilpotent coalgebras here is that the cofree objects are quite familiar, as we will soon see. In the general category of coalgebras they are large and unwieldy.

Definition 2.4. Let M be an R-module. The *tensor coalgebra over* M, denoted $T^c M$, is the coalgebra whose underlying module is

$$T^c M := R \oplus M \oplus M^{\otimes 2} \oplus \cdots$$

and whose coproduct $T^c M \to T^c M \otimes T^c M$ is given by

$$1 \mapsto 1 \otimes 1 \quad \text{and} \quad x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n)$$

where the ith summand lives in $M^{\otimes i} \otimes M^{\otimes n-i} \subset T^c M \otimes T^c M$.

For example, $x \in M$ gets mapped to $1 \otimes x + x \otimes 1$. We leave it to the reader to check that $T^c M$ is coassociative, counital, and conilpotent, with the hint that the counit is the map $T^c M \to R$ which is the identity on R and zero on higher summands.

Remark 2.5. For an R-module M, the same tensor module $R \oplus M \oplus M^{\otimes 2} \oplus \cdots$ also has a dg algebra structure given by tensor multiplication. We will often just denote $T^c M$ by TM. With this product it becomes the free associative R-algebra over M. In fact, the product and coproducts are compatible, bestowing TM with even more structure (that of a *Hopf algebra*).

Proposition 2.6. *$T^c M$ is the cofree conilpotent coalgebra over M.*

Proof. It is easy to prove the universal property, see Proposition 1.2.7, [LV12]. □

2.2. **Differential graded coalgebras.** In nature, coalgebras can often carry extra structure, just as rings or modules come with gradings.

Definition 2.7. A *differential graded* (abbreviated *dg*) coalgebra is a coalgebra (C, Δ) with

(1) a *grading*, i.e. a decomposition $C = \oplus_{n \in \mathbf{N}} C_n$ such that $\Delta(C_n) \subset \oplus_{i+j=n} C_i \otimes C_j$, and
(2) a *differential* d sending C_i to C_{i-1}, satisfying the dual identity

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta.$$

This last identity is the *Leibniz rule*; coalgebra endomorphisms satisfying it are called *coderivations*. So all differentials are coderivations. One can check that these definitions imply that the coproduct is a morphism of chain complexes. So we can think of a dg coalgebra as either (1) a chain complex with a compatible coproduct, or (2) coalgebra with compatible differential. Given a dg R-coalgebra $C = \oplus_{n \in \mathbf{N}} C_n$, we define its *suspension* sC to be C shifted up a degree, i.e. $(sC)_i = C_{i-1}$, and its *desuspension* $s^{-1}C$ to be C shifted down a degree, i.e. $(s^{-1}C)_j = C_{j+1}$. Then any $x \in C_n$ determines an element $sx \in (sC)_{n+1}$ and $s^{-1}x \in (s^{-1}C)_{n-1}$. Defining these turns out to be helpful for degree bookkeeping reasons.

The following lemma will help simplify many verifications. It tells us that to understand derivations of a free algebra, or coderivations of a cofree coalgebra, it suffices to understand them on one component.

Lemma 2.8. *Let M be an R-module. Any linear map $M \to TM$ extends uniquely to a derivation $TM \to TM$. Dually, any coderivation $T^c M \to T^c M$ is uniquely determined by its composition with the projection $T^c M \to M$.*

Proof. These follow from the more general Proposition 1.1.8 and 1.2.9 in [LV12]. ☐

The cardinal example of a coalgebra in this thesis is the following:

Example 2.9. Consider the singular chains $C_\bullet(X;\mathbf{Z})$ on a topological space X with basepoint x. They form a coaugmented dg $\mathbf{Z}$-coalgebra. For the coproduct, notice that we always have a diagonal map $x \mapsto (x,x)$, which induces a chain map $C_\bullet(X) \to C_\bullet(X \times X)$. Now there is a natural quasi-isomorphism $C_\bullet(X\times X) \to C_\bullet(X) \otimes C_\bullet(X)$, given by the decomposition of the topological simplices. Finally, $C_\bullet(X)$ is counital with the map $\epsilon : C_\bullet(X) \to \mathbf{Z}$ given by sending all 0-simplices to 1 and has a coaugmentation $\eta : \mathbf{Z} \to C_\bullet(X)$ which sends 1 to the 0-simplex at x. In Section §3, we give an explicit definition of this coproduct (called the *Alexander–Whitney* map).

2.3. **Comodules.** Continuing with our previous example of singular chains on a space X, the situation changes if we want to talk about two distinct basepoints $a, b \in X$. In this case there is no preferred choice of coaugmentation for the coalgebra $C_\bullet(X)$. To address this we need the notion of a comodule.

Definition 2.10. A *left comodule* over a unital R-coalgebra (C, Δ, ϵ) is an R-module N with a left action

$$\Delta^L : N \to C \otimes N$$

compatible with the coproduct and counit in the sense that the following diagrams commute:

$$\begin{tikzcd} N \arrow[r, "\Delta^L"] \arrow[d, "\Delta^L"'] & C \otimes N \arrow[d, "\mathrm{id} \otimes \Delta^L"] \\ C \otimes N \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes N \end{tikzcd} \qquad \begin{tikzcd} N \arrow[d, "\Delta^L"'] \arrow[dr, "\cong"] & \\ C \otimes N \arrow[r, "\epsilon \otimes \mathrm{id}"'] & R \otimes N \end{tikzcd}$$

We can similarly define a *right* comodule over C as an R-module M with the analogous right action

$$\Delta^R : M \to M \otimes C.$$

If a left and right comodule M also satisfies the following compatibility condition:

$$\begin{tikzcd} M \arrow[r, "\Delta^L"] \arrow[d, "\Delta^R"'] & C \otimes M \arrow[d, "\mathrm{id} \otimes \Delta^R"] \\ M \otimes C \arrow[r, "\Delta^L \otimes \mathrm{id}"'] & C \otimes M \otimes C \end{tikzcd}$$

then we call M a *co-bimodule*.

The simplest example of a co-bimodule over a coalgebra C is C itself, where the left and right actions are just given by the coproduct. Returning to the example of singular chains, let $C = C_\bullet(X)$ and let a be a basepoint. Then $C_\bullet(\{a\})$ is a bi-comodule over $C_\bullet(X)$ via the left and right coactions

$$C_\bullet(\{a\}) \xrightarrow{\Delta} C_\bullet(\{a\}) \otimes C_\bullet(\{a\}) \xrightarrow{\iota_a \otimes \mathrm{id}} C_\bullet(X) \otimes C_\bullet(\{a\})$$

$$C_\bullet(\{a\}) \xrightarrow{\Delta} C_\bullet(\{a\}) \otimes C_\bullet(\{a\}) \xrightarrow{\mathrm{id} \otimes \iota_a} C_\bullet(\{a\}) \otimes C_\bullet(X).$$

If $a \neq b$, we can take one of $C_\bullet(\{a\})$ and $C_\bullet(\{b\})$ to be a left comodule and the other to be a right comodule over $C_\bullet(X)$. The usefulness of these definitions will hopefully be clear as this thesis progresses.

3. (Co)simplicial methods

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{\mathrm{op}}$. This is not really how the human brain works...

[Sta26, Tag 016I]

Here we define cosimplicial objects, the totalization of a cosimplicial space, and provide some examples. We will state the Dold–Kan correspondence and its dual version, and use it to prove a useful lemma for upcoming spectral sequence computations. Finally we conclude with an exposition of the Alexander–Whitney map and its inverse.

Recall that the *simplex category*, denoted $\mathbf{\Delta}$, is the category consisting of:

(1) objects: finite nonempty totally ordered sets. We write $[n]$ for the set $\{0 < 1 < \cdots < n\}$, and
(2) morphisms: order-preserving maps, i.e. if $i \leq j$ then $f(i) \leq f(j)$.

Definition 3.1. A *cosimplicial object* in a category $\mathcal{C}$ is a functor $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$. We denote the image of $[n]$ under $X^\bullet$ by X^n. A morphism of simplicial objects $X^\bullet \to Y^\bullet$ is a natural transformation of functors, i.e. morphisms $X^n \to Y^n$ for all n that commute with morphisms in $\mathbf{\Delta}$.

We will denote the category of cosimplicial objects in $\mathcal{C}$ by $\mathrm{cs}\mathcal{C}$. In this thesis we will primarily concern ourselves with the cases where $\mathcal{C}$ is the category of topological spaces, sets, or chain complexes.

Remark 3.2. Even though simplicial objects are contravariant functors, the convention is to denote the simplices using subscripts, e.g. X_n. Conversely, even though cosimplicial objects are covariant functors, the convention is to denote the cosimplices with superscripts. Maybe the presence of the suffix co- explains this.

Recall that the morphisms in the simplex category are generated by two distinguished classes of maps. For $n \geq 1$ and $j \in [n]$, there are injections $\delta_j : [n-1] \to [n]$ where δ_j skips $j \in [n]$. For $n \geq 0$ and $j \in [n]$, there are $n+1$ surjections $\sigma_j : [n+1] \to [n]$ where σ_j sends both j and $j+1$ to $j \in [n]$. For a cosimplicial object $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$, we call the images of the δ_j *coface maps*, usually denoted d^j, and the images of the σ_j *codegeneracy maps*, usually denoted s^j. So to specify a cosimplicial object in $\mathcal{C}$ it also suffices to list a sequence of objects $X^n \in \mathcal{C}$ for $n \geq 0$, as well as coface and codegeneracy maps satisfying the following *cosimplicial identities*:

(1) If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$.
(2) If $i < j$, then $s^j \circ d^i = d^i \circ s^{j-1}$.
(3) $\mathrm{id} = s^j \circ d^j = s^j \circ d^{j+1}$.
(4) If $i > j+1$, then $s^j \circ d^i = d^{i-1} \circ s^j$.
(5) If $i \leq j$, then $s^j \circ s^i = s^i \circ s^{j+1}$.

One should think of a cosimplicial object $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$ as a diagram

$$X^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

where the rightward pointing arrows are the coface maps and the leftward pointing arrows are the codegeneracy maps. A useful heuristic is that the coface maps "duplicate a coordinate" and the codegeneracy maps "forget a coordinate." We will see this in the following examples.

Example 3.3 (Topological simplices)**.** Define the cosimplicial space $\Delta^\bullet : \mathbf{\Delta} \to \mathsf{Top}$ which sends $[n]$ to the topological n-simplex:

$$\Delta^n := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}.$$

Then the coface maps $\Delta^{n-1} \to \Delta^n$ are the inclusions of faces, where d^j is the inclusion of the face opposite the jth vertex. The codegeneracy map s^j collapses the line joining the jth and $(j+1)$th vertex. In coordinates, we have

$$d^j(x_1, \dots, x_n) = \begin{cases} (0, x_1, \dots, x_{n-1}) & j = 0 \\ (x_1, \dots, x_j, x_j, \dots, x_{n-1}) & j \in \{1, \dots, n-1\} \\ (x_1, \dots, x_{n-1}, 1) & j = n \end{cases}$$

and

$$s^j(x_1, \dots, x_n) = (x_1, \dots, \widehat{x_j}, \dots, x_n)$$

where the $\widehat{\ }$ indicates omission.

Example 3.4 (Path spaces)**.** Let X be a topological space with basepoints $a, b \in X$. Define the cosimplicial space $P^\bullet_{a,b}X$ whose cosimplices are

$$P^0_{a,b}X = \{*\}, \quad P^n_{a,b}X = X^n \text{ for } n \geq 1.$$

The coface maps $d^j : P^{n-1}_{a,b}X \to P^n_{a,b}$ are given by

$$d^j(x_1, \dots, x_{n-1}) = \begin{cases} (a, x_1, \dots, x_{n-1}) & j = 0 \\ (x_1, \dots, x_j, x_j, \dots, x_{n-1}) & j \in \{1, \dots, n-1\} \\ (x_1, \dots, x_{n-1}, b) & j = n \end{cases}$$

and the codegeneracy maps $s^j : P^{n+1}_{a,b}X \to P^n_{a,b}X$ are given by

$$s^j(x_1, \dots, x_{n+1}) = (x_1, \dots, \widehat{x_{j+1}}, \dots x_{n+1}), \quad j \in \{0, \dots, n\}.$$

We leave it to the reader to verify that the maps in both examples satisfy the cosimplicial identities. Why we refer to this example by *path spaces* will become evident in the following section.

3.1. **Totalization of cosimplicial spaces.** Cosimplicial spaces provide a useful model for many types of topological spaces, including the based path and loop spaces. This is done via a notion dual to the geometric realization of a simplicial set.

Definition 3.5. Given a cosimplicial space $X^\bullet : \mathbf{\Delta} \to \mathsf{Top}$, define the *totalization* of $X^\bullet$ to be the space of maps from the cosimplicial simplices to $X^\bullet$:

$$\mathrm{Tot}(X^\bullet) := \mathrm{Hom}_{\mathsf{csTop}}(\Delta^\bullet, X^\bullet),$$

i.e. maps $f^n : \Delta^n \to X^n$ for all $n \geq 0$ that commute with the coface and codegeneracy maps. We topologize it as a subspace of $\prod_{n \geq 0} \mathrm{Hom}(\Delta^n, X^n)$ with the compact-open topology.

So totalization gives us a functor from csTop to Top. One may ask why we need the data of a cosimplicial *space* rather than just a cosimplicial *set*, as is with the case of geometric realization of a simplicial set. It is a simple exercise to show that the totalization of a cosimplicial set (regarded as a cosimplicial discrete space) is a discrete space, which is quite uninteresting. In the next subsection we highlight some more asymmetry between simplicial and cosimplicial objects.

Let us compute the totalization of the cosimplicial space in Example 3.4. On first glance, what seems to be happening is that by iterating the face maps, we are creating finer and finer piecewise paths whose endpoints are contiguous. Indeed, after totalizing, we obtain the path space:

Proposition 3.6. $\mathrm{Tot}(P^\bullet_{a,b}X)$ *is homeomorphic to the path space* $\Omega_{a,b}X$ *of paths in* X *beginning at* a *and ending at* b.

Proof. A point of $\mathrm{Tot}(P^\bullet_{a,b}X) = \hom_{\mathsf{csTop}}(\Delta^\bullet, P^\bullet_{a,b}X)$ is a sequence of continuous maps

$$f = \{f_i : \Delta^i \to X^i\}_{i\geq 0}$$

commuting with the coface and codegeneracy maps. Fix $n \geq 2$ and $k \in \{1, \dots, n\}$. Consider the following composition of codegeneracy maps

$$\alpha_{n,k} := \underbrace{s^0 \circ \cdots \circ s^{k-2} \circ s^k \circ \cdots \circ s^{n-1}}_{n-1 \text{ maps}}$$

where we compose all the degeneracies except for s^{k-1}. This gives us a map $\Delta^n \to \Delta^1$ and likewise for $X^n \to X^1$. Then for $f = \{f_0, f_1, \dots\} \in \mathrm{Tot}(P^\bullet_{a,b}X)$, so by commutativity we obtain

$$f_1 \circ \alpha_{n,k} = \alpha_{n+1,k} \circ f_n.$$

But now the right hand side is just picking out the kth coordinate of f_n. Hence for $n \geq 2$, f_n is completely determined by f_1, so that the projection $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \mathrm{Map}(\Delta^1, X)$ given by $\{f_i\}_{i\geq 0} \mapsto f_1$ is injective.

We claim next that Φ is actually a map into $\Omega_{a,b}X \subset \mathrm{Map}(\Delta^1, X)$. The cosimplicial relations imply

$$f_1 \circ d^0 = d^0 \circ f_0 = \mathrm{const}_a, \quad f_1 \circ d^1 = d^1 \circ f_0 = \mathrm{const}_b$$

so $f_1(0) = a$ and $f_1(1) = b$ as desired.

Lastly we define an inverse to $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \Omega_{a,b}X$. For a given path $\gamma \in \Omega_{a,b}X$ consider the family of maps $\{f_i : \Delta^i \to X^i\}_{i\geq 0}$ given as follows. We let f^0 be the constant map, $f^1 = \gamma$, and for $n \geq 2$ define

$$f_n(x_1, \dots, x_n) = (\gamma(x_1), \gamma(x_n)).$$

One can check that this is an inverse to Φ. We leave it to the reader to check that the family of maps $\{f_i\}$ commutes with the coface and codegeneracies, and that Φ and its inverse are continuous maps. $\square$

We will call $P^\bullet_{a,b}X$ a cosimplicial model for the path space of X. When $a = b$ we get a cosimplicial model for the based loop space of X. It turns out that the cosimplicial space $P^\bullet_{x,x}X$ underlies the cobar construction of $C_\bullet(X)$ (though we have not defined it yet).

3.2. **Dold–Kan correspondence.** Recall that the standard Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and (non-negative) chain complexes of abelian groups. We'll restate some important results and their duals. Most of this material comes from [GJ09], Chapter III.2.

Definition 3.7. Let A be a simplicial abelian group. Its *normalized chain complex* $(NA_\bullet, \partial)$ is given by

$$NA_n = \bigcap_{i=1}^n \ker(d^i : A_n \to A_{n-1}), \quad \partial = d_0 : NA_n \to NA_{n-1}$$

and its *Moore complex* $(MA_\bullet, \partial')$ is given by

$$MA_n = A_n, \quad \partial' = \sum_{i=0}^n (-1)^i d_i : A_n \to A_{n-1}.$$

Theorem 3.8 (Dold–Kan correspondence)**.** *Let A be a simplicial abelian group. Then:*

(1) The functor $N : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ *is an equivalence of categories between simplicial abelian groups and non-negative chain complexes of abelian groups.*
(2) The inclusion of chain complexes $NA \hookrightarrow MA$ *is a chain homotopy equivalence, natural in A.*
(3) There is a functorial direct sum decomposition $MA = NA \oplus DA$*, where DA is the subcomplex generated by the images of all the degeneracy maps. Consequently, DA is acyclic.*

Now we dualize. Recall that for simplicial objects, the face maps d_i lower degree. For cosimplicial objects, the face maps d^i raise degree, so we should get an equivalence between cosimplicial abelian groups and cochain complexes of abelian groups, which is exactly what happens.

Definition 3.9. Let $C^\bullet$ be a cosimplicial abelian group. Its normalized cochain complex $(NC^\bullet, \partial)$ is given by

$$NC^n = \operatorname{coker} \bigoplus_{i=1}^{n} (d^i : C^{n-1} \to C^n), \quad \partial = d^0 : NC^n \to NC^{n+1}$$

and its Moore complex is given by

$$MC^n = C^n, \quad \partial' = \sum_{i=0}^{n+1} (-1)^i d^i : C^n \to C^{n+1}.$$

Analogously we obtain the following statements:

Theorem 3.10 (Dold–Kan correspondence, dual)**.** *Let C be a cosimplicial abelian group.*

(1) The functor $N : \mathsf{csAb} \to \mathsf{coCh}_+(\mathsf{Ab})$ is an equivalence of categories between cosimplicial abelian groups and non-negative cochain complexes of abelian groups.
(2) The quotient of cochain complexes $MC \to NC$ is a cochain homotopy equivalence, natural in C.
(3) There is a functorial direct sum decomposition $MC = NC \oplus DC$, where DC is the subcomplex generated by the images of all but one of the coface maps. Consequently, DC is acyclic.

There is another characterization of the normalized cochain complex of a cosimplicial abelian group:

Lemma 3.11. *For $C^\bullet$ a cosimplicial abelian group, its normalized chain complex is isomorphic to the complex $KC^\bullet$ with*

$$KC^n = \bigcap_{i=1}^{n-1} \ker(s^i : C^n \to C^{n-1})$$

and differential the restriction of the alternating sum of coface maps.

Proof. The proof is dual to the discussion in [GJ09] following the proof Corollary 2.3. □

Really we could have stated the previous results for simplicial objects in any abelian category. In the following discussion, let us replace "abelian group" with "R-module" for some commutative ring R, and let $X^\bullet : \mathbf{\Delta} \to \mathsf{Set}$ be a cosimplicial set. Let $R[-] : \mathsf{Set} \to \mathsf{Mod}_R$ denote the free R-module functor. Then the composition of functors $R[X^\bullet]$ yields a cosimplicial R-module. We first state a useful lemma, which was taken from a comment of Tom Goodwillie on a MathOverflow thread [Goo] related to the upcoming Proposition 3.13.

Lemma 3.12 (Goodwillie's lemma)**.** *Let $n > 0$ and choose $x, y \in X^n$. If x, y are not in the image of any coface maps $d^j : X^{n-1} \to X^n$, then $d^i x = d^j y$ implies $i = j$ and $x = y$.*

Proof. First suppose $i < j$. Then by the cosimplicial identities,$d^i x = d^j y$ implies

$$d^i s^{j-1} x = s^j d^i x = s^j d^j y = y$$

so that y is in the image of a coface map, a contradiction. The same argument for $j < i$ leaves only the possibility $i = j$. But $s^i d^i = \mathrm{id}$, implying that d^i is injective, so $x = y$. □

Proposition 3.13. *Under the Dold–Kan correspondence, the normalized cochain complex of $R[X^\bullet]$ has zero cohomology in positive degree.*

Proof. Let $NR[X^\bullet]$ be the normalized cochain complex of $R[X^\bullet]$. In each degree, we have

$$NR[X^n] = R[X^n]/D^n$$

where D^n is the free module generated by $\cup_{i=1}^n \operatorname{im} d^i$. It follows that $NR[X^n]$ is the free module on the set $U^n := X^n \setminus \cup_{i=1}^n \operatorname{im} d^i$. Let A_n be the subset of X^n consisting of elements not in the image of any coface map, and define

$$B^n = \{d^0 x : x \in A^{n-1}, d^0 x \notin \cup_{i=1}^n d^i\}.$$

We make the following two claims:

(1) For $n \geq 1$, $U^n = A^n \sqcup B^n$

Proof. Disjointness and the inclusion $A^n \sqcup B^n \subset U^n$ are clear. Conversely, let $x \in U^n$. If $x \notin A^n$, then $x = d^k(y)$ for some $y \in X^{n-1}$, in fact $k = 0$ by definition of U^n. It remains to show $y \in A^{n-1}$. If $y \notin A^{n-1}$ then $y = d^l(z)$ for some $z \in A^{n-2}$ (here if $n = 1$ this is impossible). So $x = d^0 d^l(z)$, and applying a cosimplicial identity we obtain $x = d^{l+1} d^0(z)$, contradicting the fact that $x \in U^n$. This completes the proof of the claim. □

(2) For $n \geq 2$, $B^n = \operatorname{im}(d^0 : X^{n-1} \to X^n)$.

Proof. It suffices to show that if $x \in \operatorname{im} d^0$, then $x \notin \cup_{i=1}^n \operatorname{im} d^i$. Suppose for contradiction that

$$x = d^0 a = d^j b$$

for $a \in A^{n-1}$, $b \in X^{n-1}$, and $j \geq 1$. Applying s^0 to both sides we obtain

$$a = s^0 d^0 a = s^0 d^j b = \begin{cases} b & j = 1 \\ d^{j-1} s^0 b & j > 1 \end{cases}.$$

In the first case $d^0 a = d^1 b$, contradicting Goodwillie's lemma since $a = b \in A^{n-1}$. In the second case we see that a is in the image of a coface map d^{j-1}, a contradiction. This completes the proof. □

Now we compute the coface map

$$d^0 : A^n \sqcup B^n \to A^{n+1} \sqcup B^{n+1}.$$

By the two claims, d^0 is a bijection $A^n \to B^{n+1}$. On B^n, we have $d^0 x = d^0 d^0 y = d^1 d^0 y$ for some $y \in X^{n-1}$ by cosimplicial identities, so this is zero in NC^n. It follows that

$$H^n(R[X^\bullet]) = \frac{\ker(d^0 : NR[X^n] \to NR[X^{n+1}])}{\operatorname{im}(d^0 : NR[X^{n-1}] \to NR[X^n])} = \frac{B^n}{B^n} = 0$$

completing the proof. □

Remark 3.14. The analogue of Proposition 3.13 for simplicial sets is completely false! For example, given a simplicial set $K_\bullet$, the chain complex $NR[K_\bullet]$ computes the homology of K. If these groups were all zero we would be in trouble.

3.3. **Alexander–Whitney and Eilenberg–Zilber.** In the proof of the main theorem there is a step where one needs to move between chains on a product to the product of chains. The canonical way to do so is called the Alexander–Whitney map, which we now recall.

The Alexander–Whitney map is a natural transformation of functors between the category of simplicial abelian groups and the category of chain complexes of abelian groups. It "preserves" [1] their respective monoidal structure, which we now recall. The monoidal product on sAb is given levelwise: for $A, B \in \mathsf{sAb}$, we have

$$(A \otimes B)_n = A_n \otimes B_n$$

and the face and degeneracy maps are inherited naturally. The monoidal product on the category of chain complexes is given as follows: for $X, Y \in \mathsf{Ch}_+(\mathsf{Ab})$, define

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q.$$

and the new differential $(X \otimes Y)_n \to (X \otimes Y)_{n-1}$ by the Koszul sign rule:

$$\partial_{X \otimes Y}(x \otimes y) = \partial_X x \otimes y + (-1)^{\deg x} x \otimes \partial_Y y.$$

Definition 3.15. Let $M : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ be the Moore complex functor. The *Alexander–Whitney* map $\mathcal{AW}$ is a natural transformation of the functors

$$M(- \otimes -), M(-) \otimes M(-) : \mathsf{sAb} \times \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$$

whose components are given as follows. For $A, B \in \mathsf{sAb}$ and $a \in A_n, b \in B_n$, define the map

$$\mathcal{AW}_{A,B} : M(A \otimes B) \to M(A) \otimes M(B), \quad a \otimes b \mapsto \sum_{p+q=n} \tilde{d}_p(a) \otimes d_q(b)$$

where $\tilde{d}_p$ is the face map induced by $[p] \to [n], i \mapsto i$, and d_q is the face map induced by $[q] \to [n], i \mapsto i + p$.

Proposition 3.16. *$\mathcal{AW}$ restricts to a natural transformation*

$$N(- \otimes -) \implies N(-) \otimes N(-)$$

where N is the normalized chains functor. Moreover, for $A, B \in \mathsf{sAb}$,

$$\mathcal{AW}_{A,B} : N(A \otimes B) \to N(A) \otimes N(B)$$

is a natural quasi-isomorphism.

Proof. See Section 2.5.8 of Kerodon [Lur26, Tag 00S0]. □

If we consider the simplicial abelian group of singular chains on a space, translating the formula from the definition shows that the Alexander–Whitney map corresponds to the division of a simplex into its front and back faces, and by the corollary below, gives us a coproduct on $C_\bullet(X)$. Let X, Y be topological spaces, and $C_\bullet(-)$ the singular chains functor from Top to sAb. Then $MC_\bullet(X)$ is the singular chain complex of X in the usual sense.

Corollary 3.17. *There is a natural quasi isomorphism*

$$MC_\bullet(X \times Y) \to MC_\bullet(X) \otimes MC_\bullet(Y).$$

[1] We use quotation marks here because this preservation is only up to homotopy, and is not strictly monoidal.

Proof. Recall that $C_\bullet(-)$ is the composition of the following functors:

$$\mathsf{Top} \xrightarrow{X \mapsto \mathrm{Map}(\Delta^\bullet, X)} \mathsf{sSet} \xrightarrow{A_\bullet \mapsto \mathbf{Z}[A_\bullet]} \mathsf{sAb}$$

and under this composition we get

$$X \times Y \mapsto \mathrm{Map}(\Delta^\bullet, X \times Y) = \mathrm{Map}(\Delta^\bullet, X) \times \mathrm{Map}(\Delta^\bullet, Y) \mapsto \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, X)] \otimes \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, Y)]$$

because the free abelian group functor sends Cartesian products of sets to $\mathbf{Z}$-tensor products of groups. So the simplicial abelian group $C_\bullet(X \times Y)$ is equal to $C_\bullet(X) \otimes C_\bullet(Y)$. Then, by Proposition 3.16 and the Dold–Kan correspondence we are done. □

What we have defined here is a topological Alexander–Whitney map which is a natural transformation of the functors

$$MC_\bullet(- \times -) \implies MC_\bullet(-) \otimes MC_\bullet(-) : \mathsf{Top} \times \mathsf{Top} \to \mathsf{sAb}$$

For topological spaces X, Y, denote the component map by $\mathcal{AW}_{X,Y}$. From now we will use $C_\bullet(X)$ to mean both the singular (Moore) chain complex and the simplicial abelian group, depending on context.

Corollary 3.18. *For $q \geq 0$ and a topological space X, there is a natural quasi-isomorphism of chain complexes*

$$\mathcal{AW}_q : C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}.$$

Proof. Iterate the topological Alexander–Whitney map

$$C_\bullet(X \times X^{q-1}) \xrightarrow{\mathcal{AW}_{X,X^{q-1}}} \cdots \xrightarrow{\mathrm{id}^{\otimes q-2} \otimes \mathcal{AW}_{X,X}} C_\bullet(X)^{\otimes q}.$$

Since each component is a natural quasi-isomorphism, so is the composition. □

As promised, there is also a map going the other way, called the *Eilenberg–Zilber* map, a natural transformation of functors

$$\mathcal{EZ} : M(-) \otimes M(-) \implies M(- \otimes -).$$

We are content without having an explicit formula for this map; it is slightly more finicky than the formula for Alexander–Whitney, and all we need is the following theorem:

Theorem 3.19 (Eilenberg–Zilber theorem, [EZ53])**.** *For simplicial abelian groups A, B, there are chain homotopies*

$$\mathcal{AW} \circ \mathcal{EZ} \simeq \mathrm{id}_{M(A) \otimes M(B)}, \quad \mathcal{EZ} \circ \mathcal{AW} \simeq \mathrm{id}_{M(A \otimes B)}.$$

which are natural in A, B. □

It follows that $\mathcal{EZ}$ is also a quasi-isomorphism, and these maps are inverses. Specializing to our topological setting, we obtain

Corollary 3.20. *For $q > 0$ and a topological space X, there is a natural quasi-isomorphism of chain complexes*

$$\mathcal{EZ}_q : C_\bullet(X)^{\otimes q} \to C_\bullet(X^q).$$

Moreover, $\mathcal{EZ}_q \circ \mathcal{AW}_q$ is chain homotopic to the identity on $C_\bullet(X^q)$ and $\mathcal{AW}_q \circ \mathcal{EZ}_q$ is chain homotopic to the identity on $C_\bullet(X)^{\otimes q}$. □

Remark 3.21. The proof of the Eilenberg–Zilber theorem appearing in [EZ53] does not contain any explicit formulas, making use instead of acyclic models. Explicit formulas for the maps involved can be found in [EML53], stated in terms of (p, q)-*shuffles*, which encode to the way simplices can be partitioned and combined.

4. Adams' cobar construction

In this section we define the cobar construction of a coalgebra and introduce Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ purely algebraically from the singular chains on X. We then extend the cobar construction to account for two basepoints by using comodules over a coalgebra, and prove some useful facts about the underlying homological algebra.

4.1. **Definition and Adams' theorem.** Let $(C = \overline{C} \oplus R, \Delta)$ be a dg coaugmented counital conilpotent R-coalgebra. The cobar construction is a functor from this category of coalgebras to the category of dg associative algebras. There are many references for the following definition, including Adams' original paper [Ada56]. For this first definition we follow the presentation given in [Riv22].

Definition 4.1. Let (C, Δ, ∂) be as above. The *cobar construction* on C is the dg R-algebra whose underlying algebra is the tensor algebra over the desuspension of the reduced coalgebra $\overline{C}$:

$$\mathsf{Cobar}(C) := T(s^{-1}\overline{C}) = R \oplus s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2} \oplus \cdots$$

and whose differential is given by extending the linear map

$$-s^{-1} \circ \partial \circ s^{+1} + (s^{-1} \otimes s^{-1}) \circ \overline{\Delta} \circ s^{+1} : s^{-1}\overline{C} \to s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2}$$

to all of $T(s^{-1}\overline{C})$, which yields a linear map of degree -1 from $T(s^{-1}\overline{C})$ to itself.

The definition of the differential here is valid owing to Lemma 2.8, which says that it suffices to define the differential on the component $s^{-1}\overline{C} \subset T(s^{-1}\overline{C})$. Regardless, it will be helpful to be a little more explicit. By the discussion in the previous sections, as a chain complex, $\mathsf{Cobar}(C)$ has terms

$$\begin{aligned} \mathsf{Cobar}_k(C) &= (T(s^{-1}\overline{C}))_k \\ &= \bigoplus_{q \geq 0} (s^{-1}\overline{C})_k^{\otimes q} \\ &= \bigoplus_{q \geq 0} \bigoplus_{n_1 + \cdots + n_q = k} (s^{-1}\overline{C})_{n_1} \otimes \cdots \otimes (s^{-1}\overline{C})_{n_q} \\ &= \bigoplus_{q \geq 0} \bigoplus_{n_1 + \cdots + n_q = k} \overline{C}_{n_1+1} \otimes \cdots \otimes \overline{C}_{n_q+1} \\ &= \bigoplus_{q \geq 0} \bigoplus_{m_1 + \cdots + m_q = k+q} \overline{C}_{m_1} \otimes \cdots \otimes \overline{C}_{m_q} \\ &= \bigoplus_{p+q=-k} \bigoplus_{m_1 + \cdots + m_q = -p} \overline{C}_{m_1} \otimes \cdots \otimes \overline{C}_{m_q} \end{aligned} \tag{4.1}$$

The purpose of introducing the dummy variable p at the end of this tedious indexing exercise was to suggest that we should be able to realize the cobar construction as the totalization of a second quadrant bicomplex. Indeed, in the cobar differential, we utilize both the internal differential ∂ of C with its differential graded structure as well as the coproduct Δ of C as a coalgebra.

Definition 4.2. Let (C, Δ, ∂) be as above. The *cobar bicomplex* is the bicomplex $\mathsf{Cobar}_{\bullet,\bullet}(C)$ with

$$\mathsf{Cobar}_{p,q} := \bigoplus_{m_1 + \cdots + m_q = -p} \overline{C}_{m_1} \otimes \cdots \otimes \overline{C}_{m_q}$$

and differentials

$$d_H : \mathsf{Cobar}_{p,q} \to \mathsf{Cobar}_{p+1,q}, \quad x_1 \otimes \cdots \otimes x_q \mapsto \sum_{i=1}^{q} (-1)^{\sigma(x_i)} x_1 \otimes \cdots \otimes \partial(x_i) \otimes \cdots \otimes x_q$$

$$d_V : \mathsf{Cobar}_{p,q} \to \mathsf{Cobar}_{p,q+1}, \quad x_1 \otimes \cdots \otimes x_q \mapsto \sum_{i=1}^{q} (-1)^{\sigma(x_i)} x_1 \otimes \cdots \otimes \overline{\Delta}(x_i) \otimes \cdots \otimes x_q$$

where $\sigma(x_i) = \deg x_1 + \cdots + \deg x_{i-1}$.

Proposition 4.3. *The totalization of the bicomplex* $\mathsf{Cobar}_{\bullet,\bullet}(C)$ *given by*

$$\mathrm{Tot}_k(\mathsf{Cobar}_{\bullet,\bullet}) = \bigoplus_{p+q=-k} \mathsf{Cobar}_{p,q}$$

with differential $\partial_{\mathsf{Cobar}} = d_H + (-1)^p d_V$ *on* $\mathsf{Cobar}_{\bullet,q}$, *is isomorphic as a dg algebra to* $\mathsf{Cobar}_{\bullet}(C)$.

Proof. By Equation 4.1, we have an equality of groups $\mathrm{Tot}_k(\mathsf{Cobar}_{\bullet,\bullet})$ and $\mathsf{Cobar}_k(C)$. Since d_H is of degree $(+1, 0)$ and d_V is of degree $(0, +1)$, the total differential $\partial_{\mathsf{Cobar}}$ has total degree -1. By Lemma 2.8, it suffices to show they agree on the component $s^{-1}\overline{C}$. Indeed, under the identification in Equation 4.1, we have

$$(s^{-1}\overline{C})_p = \mathsf{Cobar}_{p,1}$$

and on this component ∂_{Tot} is precisely $\partial_{\mathsf{Cobar}} = d_H - d_V$. □

Warning 4.4. We have overloaded the notation Cobar, first to refer to the functor from Definition 4.1 sending a dg coalgebra to a dg algebra, and second to refer to the functor from Definition 4.2 sending a dg coalgebra to a bicomplex. We will distinguish these using subscripts: using $\mathsf{Cobar}_\bullet$ for the first and $\mathsf{Cobar}_{\bullet,\bullet}$ for the second. But the next proposition shows that this distinction is not so important.

What is miraculous is that this purely algebraic construction encodes a good amount of topological information. This is the content of Adams' theorem:

Theorem 4.5 ([Ada56])**.** *Let* (X, x) *be a simply connected based topological space. Let* $\Omega_x X$ *be the space of loops in* X *based at* x. *Then there is a natural isomorphism*

$$H_\bullet(\mathsf{Cobar}(C_\bullet(X))) \cong H_\bullet(\Omega_x X).$$

Thus we can regard the cobar construction as the algebraic analogue of the based loop space functor. As the prefix *co-* suggests, the cobar construction is adjoint to a functor

$$\mathsf{Bar} : \mathsf{dgAlg} \to \mathsf{dgConilCoalg}$$

called the *bar construction*, which is (roughly) the algebraic version of the delooping of a space. We will briefly touch on the bar construction again in §5.

Remark 4.6. The isomorphism in Adams' theorem actually occurs at the chain level. Moreover, Rivera has shown in [Riv22] that the simply-connectedness assumption can be reduced to path-connectedness.

Remark 4.7. The name *bar* construction comes from typesetting. Pre-LaTeX, a tensor $x_1 \otimes \cdots \otimes x_n$ would be written $x_1|\cdots|x_n$. Adams attributes the name *cobar* to H. Cartan (see footnote 3 of [Ada56]).

4.2. **Two-sided version.** The cobar construction of a dg coalgebra is a special case of a more general *two-sided* cobar construction, which will allow us to talk about path spaces with differing endpoints. An analogous two-sided bar construction is well documented (for example in [Zha19], [MZZ25]). Here we give an exposition of the dual construction, which we could not find in the literature.

Definition 4.8. Let (C, Δ) be a conilpotent R-coalgebra and let (N, Δ^L) and (M, Δ^R) be left and right comodules over C, respectively. The *two-sided cobar construction on* (N, C, M) is the cosimplicial R-module

$$\mathsf{Cobar}(N, C, M)^q = M \otimes C^{\otimes q} \otimes N$$

with cofaces $d^i : \mathsf{Cobar}^{q-1} \to \mathsf{Cobar}^q$ given by

$$d^i = \begin{cases} \Delta^R \otimes \mathrm{id}^{\otimes q-1} \otimes \mathrm{id}_N & i = 0 \\ \mathrm{id}_M \otimes \mathrm{id}_C^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}_C^{\otimes q-i-1} \otimes \mathrm{id}_N & i \in \{1, \dots, q-1\} \\ \mathrm{id}_M \otimes \mathrm{id}^{\otimes q-1} \otimes \Delta^L & i = q \end{cases}$$

and codegeneracies $s^i : \mathsf{Cobar}^{q+1} \to \mathsf{Cobar}^q$ given by

$$s^i = \mathrm{id}_M \otimes \mathrm{id}^{\otimes i-1} \otimes \epsilon \otimes \mathrm{id}^{\otimes q-i+2} \otimes \mathrm{id}_N, \quad i \in \{0, \dots, q\}$$

where $\epsilon : C \to \mathbf{Z}$ is the counit.

By the Dold–Kan correspondence we can also regard this as a chain complex of R-modules. If C also has dg structure, and N, M are dg comodules over C, then then we get an internal differential on $\mathsf{Cobar}(N, C, M)$ in addition to the differential coming from the coalgebra and comodule maps. Taking these new differentials into account, we obtain the following definition, generalizing the previous discussion of the cobar bicomplex:

Definition 4.9. Let (C, ∂_C), (N, ∂_N), and (M, ∂_M) be as above. The *two-sided cobar bicomplex* has underlying bigraded R-module

$$\begin{aligned} \mathsf{Cobar}(N, C, M)_{p,q} &= (M \otimes C^{\otimes q} \otimes N)_p \\ &= \bigoplus_{a+b_1+\dots+b_q+c=-p} M_a \otimes C_{b_1} \otimes \dots \otimes C_{b_q} \otimes N_c \end{aligned}$$

with vertical differential given by the alternating sums of the coface maps

$$d_V : \mathsf{Cobar}(N, C, M)_{\bullet,q-1} \to \mathsf{Cobar}(N, C, M)_{\bullet,q}, \quad d_V = \sum_{i=0}^{q} (-1)^i d^i$$

and horizontal differential coming from the comodule and coalgebra differentials:

$$d_H : \mathsf{Cobar}(N, C, M)_{p,q} \to \mathsf{Cobar}(N, C, M)_{p-1,q}$$

$$\begin{aligned} d_H(m \otimes c_1 \otimes \dots \otimes c_q \otimes n) = {} & \partial_M(m) \otimes c_1 \otimes \dots \otimes c_q \otimes n \\ & + \sum_{i=1}^{q} (-1)^{|m|+\sigma(c_i)} m \otimes c_1 \otimes \dots \otimes \partial_C(c_i) \otimes \dots \otimes c_q \otimes n \\ & + (-1)^{|m|+\sigma(c_q)+|c_q|} m \otimes c_1 \otimes \dots \otimes c_q \otimes \partial_N(n) \end{aligned} \tag{4.2}$$

where $\sigma(x_k) = |x_1| + \dots + |x_{k-1}|$. Then totalizing with total degree $-p-q$ and differential $d_H + (-1)^p d_V$ on $\mathsf{Cobar}(N, C, M)_{\bullet,q}$ yields a dg algebra, the *two-sided cobar construction* of (N, C, M).

This lives in the second quadrant. Once again we are overloading the notation Cobar, but having multiple arguments in $\mathsf{Cobar}(-)$ indicates that we are referring to the two-sided version. Of course, this two-sided construction reduces to the one in the previous subsection, which we will sometimes refer to by the *single-sided* cobar construction. To prove this fact, we need a lemma purely about homological algebra, whose proof we include for posterity.

Lemma 4.10. *Let $f : D_{\bullet,\bullet} \to E_{\bullet,\bullet}$ be a map of second quadrant bicomplexes of abelian groups with differentials of bidegree $(+1, 0)$ and $(0, +1)$. Then*

(1) If f is a column-wise quasi-isomorphism, then the induced map on totalizations $f_* : \mathrm{Tot}\, D \to \mathrm{Tot}\, E$ is a quasi-isomorphism.

(2) If f is a row-wise quasi-isomorphism, and there exists $N \geq 0$ such that $D_{p,q} = E_{p,q} = 0$ for all $q \geq N$, then the induced map on totalizations $f_* : \mathrm{Tot}\, D \to \mathrm{Tot}\, E$ is a quasi-isomorphism.

Proof. (1) Let $D^{\leq m}_{\bullet,\bullet}$ be the quotient complex of D obtained by quotienting out the rows above m, and likewise for E. We proceed by induction on m. If $m = 0$ we are done, so suppose for $m \geq 1$ that

$$f_* : \mathrm{Tot}(D^{\leq m-1}) \to \mathrm{Tot}(E^{\leq m-1})$$

is a quasi-isomorphism.

Totalizing by taking total degree $k = -p - q$, we have the short exact sequence

$$0 \to \mathrm{Tot}(D^{\leq m-1}) \hookrightarrow \mathrm{Tot}(D^{\leq m}) \twoheadrightarrow D_{-m,\bullet}[m] \to 0$$

where $D_{-m,\bullet}[m]$ is the shifted complex

$$\cdots \to 0 \to D_{-m,0} \to D_{-m,1} \to \cdots$$

where the $D_{-m,0}$ term lies in degree m. Then by the five lemma applied to the following diagram and the corresponding diagram in the long exact sequence on homology

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tot}(D^{\leq m-1}) \arrow[r, hook] \arrow[d] & \mathrm{Tot}(D^{\leq m}) \arrow[r, two heads] \arrow[d] & D_{-m,\bullet}[m] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tot}(E^{\leq m-1}) \arrow[r, hook] & \mathrm{Tot}(E^{\leq m}) \arrow[r, two heads] & E_{-m,\bullet}[m] \arrow[r] & 0
\end{tikzcd}$$

we conclude that f_* is a quasi-isomorphism $\mathrm{Tot}(D^{\leq m}) \to \mathrm{Tot}(E^{\leq m})$. Since D is a filtered colimit of the $D^{\leq m}$, and homology and totalization commute with filtered colimits, we conclude that f_* is a quasi-isomorphism $\mathrm{Tot}(D) \to \mathrm{Tot}(E)$.

(2) We equip $\mathrm{Tot}\, D$ with a filtration inherited from the descending filtration on rows of D and E:

$$F^r \,\mathrm{Tot}_k\, D = \bigoplus_{p+q=-k,\, q\geq r} D_{p,q}$$

and likewise for E. By assumption, this filtration is finite, exhaustive and bounded below, so its associated spectral sequence converges to the homology of the totalization. The first page of the spectral sequence is given by the horizontal homology

$$D^1_{p,q} = H_p(D_{\bullet,q}), \quad E^1_{p,q} = H_p(E_{\bullet,q})$$

and by assumption f induces an isomorphism on this page. Then by the mapping lemma (5.2.4, [Wei95]), f induces an isomorphism on the ∞-pages of D and E. Then applying the five lemma inductively on the finite filtrations of $H_\bullet(\mathrm{Tot}\, D)$ and $H_\bullet(\mathrm{Tot}\, E)$ completes the argument. □

Proposition 4.11. *Let C be a conilpotent dg R-coalgebra with coaugmentation $\eta : R \to C$. Set $N = M = R$, regarded as a dg R-comodule with R concentrated in degree zero. Define the left action $\Delta^L : N \to C \otimes N$ via*

$$\begin{tikzcd}
R \arrow[d, "\eta"'] \arrow[dr, dashed, "\Delta^L"] & \\
C \arrow[r, "\cong"'] & C \otimes R
\end{tikzcd}$$

and likewise for the right action. Then $\mathrm{Tot}_\bullet\, \mathsf{Cobar}(R, C, R)$ *as above is quasi-isomorphic to the single cobar construction* $\mathsf{Cobar}_\bullet(C)$ *of Definition 4.1.*

Proof. Consider the bicomplex that is obtained by replacing each column of $\mathsf{Cobar}(R, C, R)$ with its normalized cochain complex:

$$\begin{aligned} N\,\mathsf{Cobar}_{p,q}(R,C,R) &\cong \bigcap_{i=0}^{q-1} \ker(s^i : \mathsf{Cobar}_{p,q} \to \mathsf{Cobar}_{p,q-1})_{-p} \\ &= \bigcap_{i=0}^{q-1} \ker(\mathrm{id}^i \otimes \epsilon \otimes \mathrm{id}^{\otimes q-i-1})_p \\ &= (\ker(\epsilon)^{\otimes q})_{-p} \\ &= \bigoplus_{m_1+\cdots+m_q=-p} \overline{C_{n_1}} \otimes \cdots \otimes \overline{C_{n_q}} \\ &= \mathsf{Cobar}_{p,q}(C). \end{aligned}$$

Upon checking differentials, we conclude that the normalized two-sided cobar bicomplex $N\,\mathsf{Cobar}$ is isomorphic to the one-sided cobar bicomplex on C. By Lemma 4.10, the map $\mathsf{Cobar} \to N\,\mathsf{Cobar}$ induces a quasi-isomorphism of total complexes, completing the proof. $\square$

We hope for the two-sided cobar construction to be functorial enough, i.e. it should send most quasi-isomorphisms of dg coalgebras to quasi–isomorphisms of bicomplexes. As stated, this is not always true, not even for the single-sided cobar construction. An example may be found in Proposition 2.4.3, [LV12]. There are two remedies: (1) we can impose a stronger condition on the quasi-isomorphism of coalgebras, or (2) in view of the second half of Lemma 4.10, we can ask whether the induced map on a truncated version of cobar is a quasi-isomorphism.

For (1), let C, C' be dg conilpotent R-coalgebras equipped with the coradical filtration $F_\bullet$, and let $f : C \to C'$ be a morphism of dg conilpotent R-coalgebras. Since the maps and differentials respect the coradical filtration, f descends to a map $[f] : \mathrm{gr}_\bullet C \to \mathrm{gr}_\bullet C'$ of associated gradeds, where $\mathrm{gr}_r C = F_r C / F_{r-1} C$.

Definition 4.12. If $[f]$ is a quasi-isomorphism, we say f is a *graded quasi-isomorphism*.

Lemma 4.13. *Let R be a principal ideal domain. Let N, M be left and right conilpotent dg C-comodules. Let N', M' be left and right conilpotent dg C'-comodules. Suppose*

(1) $f : C \to C'$ is a graded quasi-isomorphism with respect to the coradical filtration,
(2) $g : N \to N'$, and $h : M \to M'$ are graded comodule maps that are graded quasi-isomorphisms with respect to the coradical filtration,
(3) C, C', N, N', M, M' and their associated gradeds are flat over R.

Then the induced map

$$(g, f, h)_* : \mathrm{Tot}\,\mathsf{Cobar}(N, C, M) \to \mathrm{Tot}\,\mathsf{Cobar}(N', C', M')$$

is a quasi-isomorphism.

Proof. We imitate the strategy of the proof of the analogous Proposition 2.4.1 in [LV12], which is over a field for the one-sided cobar construction. Define a weighting on the elements of N, C, M via

$$|x| = \{\min r : x \in F_r\}$$

where F_r is the coradical filtration on the respective modules. Consider the filtration on the two-sided cobar construction given by

$$F_r\,\mathsf{Cobar}(N, C, M) := \{m \otimes c_1 \otimes \cdots \otimes c_q \otimes n : |m| + |c_1| + \cdots + |c_q| + |n| \le r\}.$$

The totalization of Cobar inherits this filtration:

$$F_r\,\mathrm{Tot}_k\,\mathsf{Cobar}(N, C, M) := \bigoplus_{p+q=-k} F_r\,\mathsf{Cobar}_{p,q}\,.$$

The filtration is bounded below and exhaustive by conilpotency, so its associated spectral sequence converges to the homology of Tot Cobar(N, C, M). Its zeroth page is given by

$$E^0_{s,t}\,\mathsf{Cobar}(N,C,M) = \frac{F_s\,\mathrm{Tot}_{s+t}\,\mathsf{Cobar}}{F_{s-1}\,\mathrm{Tot}_{s+t}\,\mathsf{Cobar}} = \bigoplus_{q\geq 0}\frac{F_s(M\otimes C^{\otimes q}\otimes N)_{s+t}}{F_{s-1}(M\otimes C^{\otimes q}\otimes N)_{s+t}}.$$

By assumption, all the modules involved are flat, so we can commute the associated graded with the tensor products to obtain

$$E^0_{s,t}\,\mathsf{Cobar}(N,C,M) = \bigoplus_{q\geq 0}\bigoplus_{a+b_1+\cdots+b_q+c=s}(\mathrm{gr}_a\,M\otimes\mathrm{gr}_{b_1}\,C\otimes\cdots\otimes\mathrm{gr}_{b_q}\otimes\mathrm{gr}_c\,N)_{s+t}.$$

Summing over t, each column of the E^0 page is therefore given by

$$E^0_{s,\bullet} = \mathsf{Cobar}(\mathrm{gr}\,N,\mathrm{gr}\,C,\mathrm{gr}\,M)^{(s)}$$

where the superscript (s) denotes the subcomplex of elements satisfying

$$|m| + |c_1| + \cdots + |c_q| + |n| = s.$$

In other words, the induced map

$$E^0(\mathsf{Cobar}(g,f,h)) : E^0\,\mathsf{Cobar}(\mathrm{gr}\,N,\mathrm{gr}\,C,\mathrm{gr}\,M)\to E^0\,\mathsf{Cobar}(\mathrm{gr}\,N,\mathrm{gr}\,C',\mathrm{gr}\,M)$$

is exactly the induced map $\mathsf{Cobar}([g],[f],[h])$. So it suffices to prove that the column-wise maps are quasi-isomorphisms in light of Lemma 4.10.

To do so, we define another filtration G on each column $E^0_{s,\bullet}$ based on tensor length, with

$$n\otimes c_1\otimes\cdots\otimes c_q\otimes m\in G_k E^0_{s,\bullet} \iff q\geq -k.$$

This is an exhaustive increasing filtration and is bounded below, so the associated spectral sequence, which we will denote by $\tilde{E}$, converges. The filtered pieces G_k/G_{k-1} consist of elements with tensor length exactly k. Because the vertical differentials induced by the coproduct increase tensor length, they are zero on $\tilde{E}^0$, so the d^0 differential solely comes from the internal differentials of $\mathrm{gr}\,N, \mathrm{gr}\,C, \mathrm{gr}\,M$, i.e. the $\tilde{E}^0$-page is just the tensor product of the complexes $\mathrm{gr}\,N\otimes(\mathrm{gr}\,C)^{\otimes q}\otimes\mathrm{gr}\,M$. By flatness and the Künneth formula, we obtain

$$\tilde{E}^1(C) = H_\bullet(\mathrm{gr}\,N)\otimes H_\bullet(\mathrm{gr}\,C)^{\otimes q}\otimes H_\bullet(\mathrm{gr}\,M).$$

By assumption, $([g],[f],[h])$ induces an isomorphism

$$\tilde{E}^1(C)\to\tilde{E}^1(C')$$

and consequently on the columns of the original bicomplex $E^0_{s,\bullet}$, completing the proof. □

In practice, however, the notion of a graded quasi-isomorphism can be quite restrictive. For the purposes of this thesis, we will only need a truncated version of the cobar construction, in which quasi-isomorphisms of coalgebras will always induce quasi-isomorphisms. Let us make this precise. Let $\mathsf{Cobar}^{>n}$ denote the subcomplex of Cobar obtained by setting all rows below n to zero, and let $\mathsf{Cobar}^{\leq n}$ be the quotient bicomplex $\mathsf{Cobar}\,/\,\mathsf{Cobar}^{>n}$. Then $\mathsf{Cobar}^{\leq n}$ is a second quadrant bicomplex satisfying the assumptions of the second half of Lemma 4.13. Therefore, we obtain the following corollary:

Corollary 4.14. *Let R be a principal ideal domain. Let N, M be left and right conilpotent dg C-comodules. Let N', M' be left and right conilpotent dg C'-comodules. Suppose*

(1) $f : C\to C'$, $g : N\to N'$, and $h : M\to M'$ are quasi-isomorphisms,
(2) g, h are (left/right)-comodule maps,
(3) C, C', N, N', M, M' are flat over R.

Then the induced map on truncated two-sided cobar constructions

$$(g, f, h)_* : \mathsf{Cobar}^{\leq n}(N, C, M) \to \mathsf{Cobar}^{\leq n}(N', C', M')$$

induces a quasi-isomorphism on totalizations.

Proof. By flatness and the Künneth formula, the induced map on row q, given by

$$(g, f, h)_* = g \otimes f^{\otimes q} \otimes h$$

is a quasi-isomorphism. Applying the second half of Lemma 4.10 completes the proof. □

The scenario in which this applies is if we want to use normalized singular chains instead of singular chains inside of the cobar construction. Note that this is different to what we did in Proposition 4.11, where we normalized each *column* of the two-sided cobar complex, instead of replacing the inputs with their normalized counterparts.

5. Homology and the fundamental group

Better to sink beneath the shock
Than moulder piecemeal on the rock!
—Lord Byron, *The Giaour*

One can use homology and the cobar construction to approximate the fundamental group:

Theorem 5.1 ([Sta75], Theorem 5.4). *Let R be a ring and Y a simplicial set with $Y_0 = \{*\}$. Let $\mathcal{I}$ be the augmentation ideal of the group ring $R\pi_1(|Y|, *)$. Let E be the spectral sequence associated to the one-sided cobar bicomplex on the simplicial chain complex[2] associated to Y, with coaugmentation the inclusion of the 0-simplex and filtration by tensor length. Then there is an isomorphism*

$$E^{\infty}_{-p,p} \to \mathcal{I}^p / \mathcal{I}^{p+1}.$$

A dual version of this theorem with cohomology and the bar construction can be found in [Gad23], Theorem 1.2(3). The setting in which this is relevant for us is when $Y = \mathrm{Sing}(X)$ for X a topological space. The goal of this section will be to prove a version of this theorem for the two-sided cobar construction, which we will also do in the setting of simplicial sets.

Recall that the two-sided cobar functor takes in as input a dg conilpotent coalgebra and two dg comodules over it. In particular, if we begin with a connected simplicial set Y, its normalized simplicial chain complex $N_\bullet(Y)$ is a conilpotent dg algebra, and under this correspondence any simplicial subset $Y' \subset Y$ becomes a bicomodule over $N_\bullet(Y)$. Let us briefly recall/define for our purposes what the fundamental group and groupoid of Y are. To avoid technicalities about simplicial sets, we simply define them to be the corresponding notions of fundamental group(oid) for the geometric realization $|Y|$. For all purposes this is enough. [3] Recall that Y is *connected* if $\pi_0(Y) = \{*\}$, i.e. between any two 0-simplices there exist a sequences of contiguous 1-simplices connecting them. Equivalently, the geometric realization $|Y|$ is connected/path-connected.

Definition 5.2. The *fundamental groupoid* of a connected simplicial set Y is the full subcategory π_Y of $\pi_{|Y|}$ whose objects are the zero simplices. For a given 0-simplex $y \in Y_0$, the *fundamental group* of (Y, y) is the group of morphisms $\pi_1(Y, y) := \pi_Y(y, y) = \pi_{|Y|}(y, y)$.

[2] Compose $Y : \Delta^{\mathrm{op}} \to \mathsf{Set}$ with the free R-module functor, then use Dold–Kan to obtain the corresponding Moore complex. This is a dg coalgebra whose comultiplication is given by the Alexander–Whitney map, same as it is with singular chains on a topological space.

[3] The technical simplicial set definition of fundamental groupoid π_Y is category-equivalent to what we define here.

Now regard $N_\bullet(\{a\})$ as a right $N_\bullet(Y)$-comodule via the right comodule action

$$\Delta^R : N_\bullet(a) \xrightarrow{\mathcal{AW}} N_\bullet(a) \otimes N_\bullet(a) \xrightarrow{\mathrm{id}\otimes \iota_a} N_\bullet(a) \otimes N_\bullet(Y)$$

and likewise for $N_\bullet(\{b\})$ as a left $N_\bullet(X)$-comodule.

Theorem 5.3. *Let Y be a connected simplicial set and choose $a, b \in Y_0$. Let $N_\bullet(Y)$ be the normalized chain complex associated to Y, and let $N_\bullet(a), N_\bullet(b)$ be the normalized chain complexes associated to the constant subsimplicial sets at a and b respectively. Let $\mathcal{I}$ be the augmentation ideal of the group ring $\mathbf{Z}\pi_1(Y, a)$. Let E be the spectral sequence associated to the two-sided cobar construction*

$$\mathsf{Cobar}(b, Y, a) := \mathsf{Cobar}(N_\bullet(b), N_\bullet(Y), N_\bullet(a))$$

where the filtration is given by tensor length. Then there is an isomorphism

$$E^\infty_{-p,p} \to \mathbf{Z}\pi_Y(a,b)\mathcal{I}^p / \mathbf{Z}\pi_Y(a,b)\mathcal{I}^{p+1}.$$

More precisely, for any $a, b \in Y_0$ we will construct a map

$$H_0(\mathsf{Cobar}^{\leq n}(b, Y, a)) \to \mathbf{Z}\pi_Y(a,b)/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^{p+1}$$

and after some spectral sequence algebra we will deduce the theorem.

5.1. **Stallings' proof.** We will imitate Stallings' strategy. He relies on the following result of Brown:

Theorem 5.4 (Theorem 10.2, [Bro59]). *Let A be a dga R-algebra. Suppose $A_0 = R$ and $H_1(A) = 0$. Then there is a chain equivalence $\mathsf{Cobar}(\mathsf{Bar}(A)) \to A$.* □

This is an intermediate step, and because we don't discuss the bar construction again, we refer the reader to sources such as [Mil67], Chapter 16 of [Bro82], or Chapter 1.5 of [May99]. In our case we specialize to the case where G is a group and A is the chain complex consisting of the integral group ring $\mathbf{Z}G$ concentrated in degree zero. Then it is a fact that the bar construction on $\mathbf{Z}G$ is isomorphic to singular chains on the classifying space BG.

Proposition 5.5. *Let G be a group and BG its classifying space. There is a chain equivalence*

$$\mathsf{Cobar}(C_\bullet(BG)) \to \mathbf{Z}G$$

and consequently for any simplicial set Y with $Y_0 = \{*\}$, there is an isomorphism

$$H_0\,\mathsf{Cobar}(C_\bullet(Y)) \to \mathbf{Z}\pi_1(Y, *).$$

Proof. For the first map, apply Brown's theorem. The second isomorphism follows by Corollary 5.3 of [Sta75], which says that the map of Y into $B\pi_1(Y, a)$ (as a simplicial set) induces an isomorphism on zeroth homology of cobar constructions. □

Returning to Stallings' theorem, he specifically shows that the map

$$\mathrm{Tot}_0(\mathsf{Cobar}(C_\bullet(Y))) \to \mathbf{Z}\pi_1(Y, *)$$
$$\sigma_1 \otimes \cdots \otimes \sigma_q \mapsto (1_a - \sigma_1) \cdot \cdots \cdot (1_a - \sigma_q)$$

descends to an isomorphism on zeroth homology. Then it is clear that the tensor length truncations of Cobar correspond to the $\mathcal{I}$-adic filtration on the fundamental group ring, since the latter is generated by elements of the form $1 - g$. The natural generalization of Stallings' map to the two-sided case

$$\mathrm{Tot}_0(\mathsf{Cobar}(b, Y, a)) \to \mathbf{Z}\pi_Y(a, b)$$
$$1_b \otimes \sigma_1 \otimes \cdots \otimes \sigma_q \otimes 1_a \mapsto (1 - \sigma_1) \cdot \cdots \cdot (1 - \sigma_q)$$

runs into a few issues. First, it is unclear what the multiplication on the right hand side means. In the fundamental group ring the multiplication is given by path concatenation, since all the σ_i are loops with the same endpoints, but once we have different endpoints this becomes ill-defined. Secondly, recall that $\mathrm{Tot}_0\,\mathsf{Cobar}(b, Y, a)$ is generated by

$$N_\bullet(b) \otimes \underbrace{N_\bullet(Y) \otimes \cdots \otimes N_\bullet(Y)}_{q \text{ times}} \otimes N_\bullet(a), \quad \text{internal degrees summing to } q$$

so one could, for example, have cycles in $\mathrm{Tot}_0(\mathsf{Cobar}(b, Y, a))$ coming from

$$N_0(b) \otimes N_0(Y) \otimes N_2(Y) \otimes N_0(a).$$

So it is unclear how Stallings' map generalizes, since we have only defined it on 1-chains. Moreover, if we have multiple 0-simplices, then differences of 0-simplices now generate the kernel of the counit $N_0(Y) \to \mathbf{Z}$, so not all internal degree 0 tensor factors vanish. The upcoming definitions will help us address these problems.

5.2. **Two-sided isomorphism.** What we want to do is to be able to multiply the morphisms in π_Y regardless of whether their endpoints match up or not. We will simply concatenate them if the endpoints match, and make the product zero if they don't. To do so we will regard π_Y as a subset of a certain ring in the following manner.

Definition 5.6. For any small category $\mathcal{C}$ and ring R, the *category algebra* $R[\mathcal{C}]$ is the ring (possibly without multiplicative identity) whose underlying set is the free R-module on the set of all morphisms in $\mathcal{C}$. The addition on $R[\mathcal{C}]$ is given by the free module addition, and the multiplication of two morphisms is given by their composition, if they are composable, and zero otherwise.

If $\mathcal{C}$ has only finitely many objects, then the sum of all identity objects serves as the multiplicative identity. This also makes $R[\mathcal{C}]$ into an R-algebra. In our case, let Y be our simplicial set, and form the integral category algebra $\mathbf{Z}[\pi_Y]$ on its fundamental groupoid. An element of $\mathbf{Z}[\Pi_Y]$ is some $\mathbf{Z}$-linear combination of paths between 0-simplices, and multiplication of two homotopy classes of paths σ and τ is given by concatenation if the endpoint of σ is equal to the starting point of τ, and zero otherwise. By regarding the morphisms in $\mathbf{Z}\pi_Y(a,b)$ as a subset of $\mathbf{Z}[\pi_Y]$, we can multiply them. Indeed, in the case with a single 0-simplex, the multiplication reduces to path concatenation in the fundamental group ring.

The way we will prove this two-sided generalization of Stallings' theorem is to reduce it to his case where the simplicial set has only a single 0-simplex. To do so, recall that we can always choose a maximal spanning tree $T \subset Y$ which contains all 0-simplices[4]. Because trees are contractible, the quotient $Y \to Y/T$ is a homotopy equivalence, and the resultant simplicial set Y/T has only a single 0-simplex $\tilde{a}$. Let $t_{x,y}$ denote the unique path in T joining two 0-simplices x and y. Then $t_{x,y} \cdot t_{y,z} = t_{x,z}$, and $t_{x,x} = 1_a$. Given a 1-simplex σ_i of Y and fixed 0-simplex a, define

$$T_a(\sigma_i) = t_{a,d_0\sigma_i} \cdot \sigma_i \cdot t_{d_1\sigma_i,a},$$

which is an element of $\mathbf{Z}\pi_1(Y,a)$. Lastly let $T(\sigma_i)$ be the unique homotopy class in T of σ_i, which by our notation above is $t_{d_0\sigma_i,d_1\sigma_i}$. Though we tried our best, we could not avoid making a choice of tree here. Following this chapter's epigraph, let us march on and state the map regardless.

Proposition 5.7. *For $a, b \in Y_0$, the map*

$$\varphi_{a,b} : \mathrm{Tot}_0\,\mathsf{Cobar}^{\leq n}(b, Y, a) \to \mathbf{Z}\pi_Y(a,b)/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^{n+1}$$

defined by

$$\varphi_{a,b}(1_b \otimes \sigma_1 \otimes \cdots \otimes \sigma_q \otimes 1_a) = 1_b \cdot \Big(\prod_{i=1}^{q} (T(\sigma_i) - \sigma_i)\Big) \cdot 1_a$$

[4]This follows by Zorn's lemma applied to the collection of subgraphs of the 1-skeleton which are trees. It turns out also to imply Zorn's lemma! See, for example, Serre's aptly named book *Trees*, §2.3 [Ser80].

on such generators consisting of non-degenerate 1-simplices, and defined by zero otherwise, descends to an isomorphism on zeroth homology. On the right-hand side, the multiplication comes from the category algebra $\mathbf{Z}[\pi_Y]$.

Proof. We need to check that $\varphi_{a,b}$ factors through the subcomplex $\mathsf{Cobar}^{>n}$ and that it descends to homology. For the first point, take an element $\sigma = 1_b \otimes \sigma_1 \otimes \cdots \otimes \sigma_q \otimes 1_a \in \operatorname{Tot}_0 \mathsf{Cobar}(b, Y, a)$. Note first that by rewriting σ_i as $t_{d_0\sigma_i,a} \cdot T_a(\sigma_i) \cdot t_{a,d_1\sigma_i}$, we get

$$\begin{aligned} T(\sigma_i) - \sigma_i &= t_{d_0\sigma_i,d_1\sigma_i} - t_{d_0\sigma_i,a} \cdot T_a(\sigma_i) \cdot t_{a,d_1\sigma_i} \\ &= t_{d_0\sigma_i,a} \cdot (1_a - T_a(\sigma_i)) \cdot t_{a,d_1\sigma_i} \end{aligned}$$

and now this middle factor belongs to $\mathcal{I} \subset \mathbf{Z}\pi_1(Y, a)$. We may now assume the σ_i assemble into a contiguous path between a and b, for otherwise they vanish on $\varphi_{a,b}$ under the multiplication in the category algebra. We then obtain:

$$\begin{aligned} \varphi_{a,b}(\sigma) &= 1_b \cdot \Big(\prod_{i=1}^{q} T(\sigma_i) - \sigma_i\Big) \cdot 1_a \\ &= 1_b \cdot t_{d_0\sigma_1,a} \cdot (1_a - T_a(\sigma_1)) \cdot t_{a,d_1\sigma_1} \cdot \cdots \cdot t_{d_0\sigma_q,a} \cdot (1_a - T_a(\sigma_q)) \cdot t_{a,d_1\sigma_q} \cdot 1_a \\ &= t_{b,a} \cdot \prod_{i=1}^{q} (1_a - T_a(\sigma_i)) \in \mathbf{Z}\pi_Y(a, b)\mathcal{I}^q. \end{aligned}$$

Hence $\varphi_{a,b}$ factors through $\mathsf{Cobar}^{>n}$.

For the second point, by definition, $\operatorname{Tot}_1 \mathsf{Cobar}$ necessarily contains a tensor factor coming from $N_2(Y)$, say Γ. Computing the two-sided cobar differential, we have

$$\begin{aligned} \varphi_{a,b}(\partial\Gamma) &= (1 - d_0\Gamma) - (1 - d_1\Gamma) + (1 - d_2\Gamma) \\ &= 1 - d_0\Gamma + d_1\Gamma - d_2\Gamma \\ \varphi_{a,b}(\overline{\Delta}\Gamma) &= (1 - d_2\Gamma)(1 - d_0\Gamma) \\ &= 1 - d_0\Gamma - d_2\Gamma + d_2\Gamma * d_0\Gamma \end{aligned}$$

so that

$$\varphi_{a,b} \circ \partial_{\mathsf{Cobar}}(1_a \otimes \Gamma \otimes 1_b) = \varphi_{a,b}(\partial\Gamma - \overline{\Delta}(\Gamma)) = [d_1\Gamma] - [d_2\Gamma * d_0\Gamma] = 0$$

since $d_2\Gamma * d_0\Gamma$ is homotopic to $d_1\Gamma$. Thus $\varphi_{a,b}$ descends to a well defined map on zeroth homology.

The remainder of the proof relies on the aforementioned comparison with Stallings' theorem. The quotient of the tree induces a quasi-isomorphism

$$q : N_\bullet(Y) \to N_\bullet(Y/T).$$

Now let D denote the two-sided cobar bicomplex $\mathsf{Cobar}(b, Y, a)$ and $D^{>n}$, $D^{\leq n}$ be the previous subcomplex and quotient. Then q induces a map from D onto the two-sided cobar construction on

$$(N_\bullet(\tilde{a}), N_\bullet(Y/T), N_\bullet(\tilde{a})). \tag{5.1}$$

By Lemma 4.10 and Proposition 4.11, the two-sided cobar construction on (5.1) is quasi-isomorphic after totalization to the single-sided cobar construction $\mathsf{Cobar}(C_\bullet(Y/T))$ with coaugmentation the inclusion of $*$. Let E, $E^{>n}$, $E^{\leq n}$ denote the corresponding bicomplexes for $\mathsf{Cobar}(C_\bullet(Y/T))$. This gives us a quasi-isomorphism $\operatorname{Tot} D^{\leq n} \to E^{\leq n}$. Consider now the following diagram:

$$\begin{tikzcd} H_0(D^{\leq n}) \arrow[r, "q_*"] \arrow[d, dashed, "\varphi_{a,b}"'] & H_0(E^{\leq n}) \arrow[d, "5.1"] \\ \mathbf{Z}\pi_Y(a,b)/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^{n+1} \arrow[r, "q_*"'] & \mathbf{Z}\pi_1(Y,\tilde{a})/\mathcal{I}^{n+1} \end{tikzcd}$$

By the previous discussion, the top arrow is an isomorphism. The right arrow is an isomorphism by Stallings' theorem; his map is just $\varphi_{\tilde{a},\tilde{a}}$. The bottom map is left concatenation with the element $t_{a,b}$, the pasing to the quotient. Because T is a maximal tree this is an isomorphism. It remains to show that $\varphi_{a,b}$ commutes with these maps. We have

$$q_* \circ \varphi_{a,b}(\sigma) = q_*(t_{b,a} \cdot \prod_{i=1}^{q}(1_a - T_a(\sigma_i))) = 1_{\tilde{a}} \cdot \prod_{i=1}^{q}(1_{\tilde{a}} - q_* T_a(\sigma_i)).$$

On the other hand, since q_* collapses any path in T, we have

$$q_*\sigma_i = q_*(t_{d_0\sigma_i,a} \cdot T_a(\sigma_i) \cdot t_{a,d_1\sigma_i}) = q_* T_a(\sigma_i),$$

so that

$$\begin{aligned}
\varphi_{\tilde{a},\tilde{a}} \circ q_*(\sigma) &= 1_{\tilde{a}} \cdot (\prod_{i=1}^{q} T(q_*\sigma_i) - q_*\sigma_i) \cdot 1_{\tilde{a}} \\
&= 1_{\tilde{a}} \cdot (\prod_{i=1}^{q} 1_{\tilde{a}} - q_* T_a(\sigma_i)) \cdot 1_{\tilde{a}} \\
&= 1_{\tilde{a}} \cdot (\prod_{i=1}^{q} 1_{\tilde{a}} - q_* T_a(\sigma_i))
\end{aligned}$$

showing commutativity and concluding the proof. □

Proof of Theorem 5.3. The exact sequence $0 \to D^{>n} \to D \to D^{\leq n} \to 0$ induces a long exact sequence

$$\cdots \to H_0(D^{>n}) \to H_0(D) \to H_0(D^{\leq n}) \to H_{-1}(D^{>n}) \to \cdots.$$

We claim that the homology groups of $D^{>n}$ and D are zero in negative degree. While this is immediate for $E^{>n}$ and E, it is not *a priori* obvious here due to the second issue we mentioned earlier in this section, which said that terms in $\mathrm{Tot}_{-1}(D)$ can be nonzero. Let us work with the normalized two-sided cobar construction, which is quasi-isomorphic after totalizing to D. Its internal tensor factors of $N_\bullet(Y)$ are replaced by their reduced counterparts $\overline{N_\bullet(Y)}$ by the same computation as in Proposition 4.11. The elements of total degree k therefore belong to

$$(N_\bullet(b) \otimes \overline{N_\bullet(Y)}^{\otimes q} \otimes N_\bullet(a))_{-p}$$

where $k = -p - q$. Filtering this by tensor length (which is exhaustive), we obtain a spectral sequence

$$E^1_{p,q} = H_{-p}(N_\bullet(b) \otimes \overline{N_\bullet(Y)}^{\otimes q} \otimes N_\bullet(a)).$$

Since Y is connected, $H_k(\overline{N_\bullet(Y)}) = 0$ for $k \leq 0$, so by the Künneth formula, $H_k(\overline{N_\bullet(Y)}^{\otimes q}) = 0$ when $k < q$. Hence the E^1 page is zero under the diagonal, when $-p < q$. Thus when $k = -p - q < 0$, the total complex has no homology in degree k. The same argument applies for the subcomplex $D^{>n}$. We consequently get a surjection

$$H_0(D) \to H_0(D^{\leq n})$$

whose kernel is $H_0(D^{>n})$, i.e. the $(n+1)$th filtered piece of $H_0(D)$. Thus $H_0(D)/F_{n+1}H_0(D) \cong H_0(D^{\leq n})$. We conclude on the ∞-page of the spectral sequence associated to D that

$$\begin{aligned}
D^\infty_{-n,n} &\cong F_n H_0(D)/F_{n+1}H_0(D) \\
&\cong \ker(H_0(D^{\leq n}) \to H_0(D^{\leq n-1})) \\
&\cong \ker(\mathbf{Z}\pi_Y(a,b)/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^{n+1} \to \mathbf{Z}\pi_Y(a,b)/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^n) \\
&= \mathbf{Z}\pi_Y(a,b)\mathcal{I}^n/\mathbf{Z}\pi_Y(a,b)\mathcal{I}^{n+1}
\end{aligned}$$

where the third isomorphism is by Proposition 5.7, completing the proof. □

This proof addresses the second issue we encountered earlier. The terms of Tot_0 Cobar containing an internal tensor factor of $C_l(Y)$ for $l \geq 2$ get sent to zero. This makes sense, since π_1 and $\pi_X(a,b)$ only contain information about the 1-simplices. Let us conclude by translating these two results back into the language of topological spaces for later.

Corollary 5.8. *Let X be a path-connected topological space with two basepoints a, b. Then there are isomorphisms*

$$H_0 \,\mathsf{Cobar}^{\leq n}(N_\bullet(b), N_\bullet(X), N_\bullet(a)) \to \mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{n+1},$$

$$E^\infty_{-p,p} \to \mathbf{Z}\pi_X(a,b)\mathcal{I}^p/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{p+1}$$

where E is the spectral sequence associated to $\mathsf{Cobar}(N_\bullet(b), N_\bullet(X), N_\bullet(a))$ *with the tensor-length filtration.*

Proof. Apply Theorem 5.3 and Proposition 5.7 with $Y = \mathrm{Sing}(X)$, and note that π_Y and π_X are naturally isomorphic. ☐

6. Proof of main theorem

We are finally ready to present the proof of Theorem 1.1. Recall that X is a path-connected topological space with basepoints a, b. Our strategy will be to consider the other spectral sequence associated to the two-sided cobar construction. To do so, we will first build a spectral sequence starting from the cosimplicial path space $P^\bullet_{a,b}X$ of Example 3.4. We will show that it computes the group $H_n(X^n, X(n)^a_b)$, and that it arises from the two-sided cobar construction. Then we will conclude by the previous discussion relating the cobar construction to the fundamental group.

6.1. **Building another spectral sequence.** Denote $P^\bullet_{a,b}X$ by $P^\bullet$ for simplicity. Notice first that the union of the images of the coface maps $d^i : P^{n-1} \to P^n$ is exactly the subspace $X(n)^a_b$. This is a good sign. Now consider the second-quadrant bicomplex $\mathscr{C}_{\bullet,\bullet}$ which is obtained by applying singular chains to $P^\bullet$:

$$\mathscr{C}_{p,q} = C_{-p}(P^q)$$

with "horizontal" differential $d^{\mathscr{C}}_H : C_{-p}(P^q) \to C_{-(p+1)}(P^q)$ given by the singular boundary map, and "vertical" differential $d^{\mathscr{C}}_V : C_p(P^q) \to C_p(P^{q+1})$ given by the differential of the Moore complex of the cosimplicial abelian group $C_p(P^\bullet)$. Each column $\mathscr{C}_{p,\bullet}$ is the Moore complex of $C_p(P^\bullet)$. Consider now the bicomplex $\mathscr{N}_{\bullet,\bullet}$ whose columns are instead the normalized cochain complexes of $C_p(P^\bullet)$. By Dold–Kan, and Lemma 4.10, the map $\mathscr{C} \twoheadrightarrow \mathscr{N}$ is a quasi-isomorphism after totalizing. Let us spell out what the groups in $\mathscr{N}$ actually are. Following the definition, we have

$$\mathscr{N}_{p,q} = \mathrm{coker} \bigoplus_{i=1}^{n} (d^i_* : C_p(X^{n-1}) \to C_p(X^n)) = C_p(X^n)/\langle \sum_{i=1}^{n} \mathrm{im}(d^i)_* \rangle = C_p(X^n, X(n)_b).$$

Here $X(n)_b$ denotes the subspace of X^n consisting of those $(x_1, \ldots, x_n)$ where $x_i = x_{i+1}$ for $i \in \{1, \ldots, n-1\}$ or $x_n = b$. Moreover, the vertical differential $\mathscr{N}_{p,q} \to \mathscr{N}_{p,q+1}$ is induced by the zeroth coface map d^0. Consider now the bicomplex obtained from $\mathscr{N}$ by quotienting out the rows above $q = n$, which we will denote by $\mathscr{N}^{\leq n}$.

Proposition 6.1. *Filtering $\mathscr{N}^{\leq n}$ by the lower index, we obtain a spectral sequence with terms*

$$E^2_{-n,n} = H_n(X^n, X(n)^a_b), \quad E^2_{0,0} = \begin{cases} \mathbf{Z} & a = b \\ 0 & a \neq b. \end{cases}$$

collapsing on the second page and converging to $H_\bullet \mathscr{N}^{\leq n}$. Moreover, all rows $E^2_{\bullet,q}$ are zero for $q \neq 0, n$.

Proof. The filtration is bounded below and exhaustive, so the spectral sequence converges. To compute the E^2-page, let us begin on E^0, whose $-p$th column is the following normalized cochain complex (beginning below in degree $q = 0$):

$$C_p(*) \to C_p(X, \{b\}) \to \cdots \to C_p(X^n, X(n)_b) \to 0 \to \cdots$$

of the Moore complex

$$C_p(*) \to C_p(X) \to \cdots \to C_p(X^n) \to 0 \to \cdots. \tag{6.1}$$

By Dold–Kan, these complexes have the same homology. But now (6.1) is just the cochain complex obtained by taking the free abelian group on the cosimplicial set $[q] \mapsto \mathrm{Map}(\Delta^p, P^q)$, with terms $q > n$ set to zero. Therefore, by Proposition 3.13, we obtain $E^1_{-p,q} = 0$ for $q \neq 0, n$. For the remaining nonzero rows $q = n$ and $q = 0$, we have

$$E^1_{-p,n} = C_p(X^n, X(n)_b)/\operatorname{im}(d^0) = C_p(X^n, X(n)^a_b)$$

$$E^1_{-p,0} = \ker(d^0 : C_p(*) \to C_p(X, \{b\})) = \begin{cases} \mathbf{Z} & p = 0, a = b \\ 0 & \text{otherwise} \end{cases}.$$

On the E^1-page, the differential is now induced by singular boundary map. Therefore, after taking homology we obtain

$$E^2_{-p,n} = H_p(X^n, X(n)^a_b).$$

And on the other possibly nonzero row $q = 0$, we have

$$E^2_{-p,0} = \begin{cases} H_p(\mathbf{Z} \leftarrow 0 \leftarrow \cdots) & a = b \\ 0 & a \neq b \end{cases}$$

as desired.

At this point, the only nonzero rows are $q = n$ and possibly $q = 0$. The differentials on the pages $E^{\geq 2}$ are all zero, except possibly on page E^{n+1}, where we have a differential d_{n+1} of bidegree $(n+1, -n)$ from $E^{n+1}_{-n-1,n} = H_{n+1}(X^n, X(n)^a_b)$ to $E^{n+1}_{0,0} = \mathbf{Z}$. To see that $d_{n+1} = 0$, consider the map of pairs $f : (X, x) \to (*, *)$. Repeating the constructions, we get a spectral sequence D that computes $H_n(*^n, *(n)^*_*) = 0$. By functoriality, f induces a morphism of spectral sequences. Specializing to the $n + 1$th page, we have a commutative square

$$\begin{tikzcd} H_{n+1}(X^n, X(n)^x_x) \arrow[r, equal] & E^{n+1}_{-n-1,n} \arrow[r, "d^E_{n+1}"] \arrow[d, "f_*"'] & E^{n+1}_{0,0} \arrow[r, equal] \arrow[d, "f_*"] & \mathbf{Z} \\ H_{n+1}(*, *) = 0 \arrow[r, equal] & D^{n+1}_{-n-1,n} \arrow[r, "d^D_{n+1}"'] & D^{n+1}_{0,0} \arrow[r, equal] & \mathbf{Z} \end{tikzcd}$$

But now $f_* : \mathbf{Z} \to \mathbf{Z}$ is the identity since it is the induced map $H_0(X, x) \to H_0(*, *)$, forcing $d^E_{n+1} = 0$, so $E^2 = E^\infty$ as desired. ◻

Corollary 6.2. *We have a surjection*

$$H_0(\mathrm{Tot}(\mathscr{N}^{\leq n})) \twoheadrightarrow H_n(X^n, X(n)^a_b)$$

whose kernel is $\mathbf{Z}$ *if* $a = b$*, and trivial otherwise.*

Proof. Follows from convergence of the spectral sequence in Proposition 6.1. ◻

6.2. **Four bicomplexes.** It remains to compute H_0 of this totalization. For this we will return to our original bicomplex $\mathscr{C}$ and show that its truncated totalization is quasi-isomorphic to the totalization of the two-sided cobar bicomplex. Instead of normalizing the columns, let us now consider the rows $\mathscr{C}_{\bullet,q}$, which are the singular chain complexes $C_\bullet(X^q)$. Then we can apply the topological Alexander–Whitney map on each row:

$$\mathscr{C}_{-\bullet,q} = C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}$$

to obtain a new bicomplex, which we will call $\mathscr{D}$. By definition of tensor product of chain complexes, $\mathscr{D}$ has a bigrading given by

$$\mathscr{D}_{-p,q} = (C_\bullet(X)^{\otimes q})_p = \bigoplus_{n_1+\cdots+n_q=p} C_{n_1}(X)\otimes\cdots\otimes C_{n_q}(X).$$

Let $\mathscr{C}^{\leq n}$ and $\mathscr{D}^{\leq n}$ be the bicomplexes obtained from $\mathscr{C}$ and $\mathscr{D}$ by quotienting all rows above $q = n$.

Lemma 6.3. *The map $\mathscr{C}^{\leq n} \to \mathscr{D}^{\leq n}$ induced by Alexander–Whitney is a quasi-isomorphism of totalizations, with inverse quasi-isomorphism induced by the Eilenberg–Zilber map.*

Proof. Combine Corollary 3.18, Theorem 3.19, and Lemma 4.10. □

Lemma 6.4. *There is an isomorphism*

$$H_\bullet \operatorname{\mathsf{Cobar}}^{\leq n}(N_\bullet(b), N_\bullet(X), N_\bullet(a)) \cong H_\bullet \mathscr{D}^{\leq n}.$$

Proof. Denote the left-hand side by Cobar. By Lemma 4.10, it suffices to exhibit a such a map from Cobar to $N\mathscr{D}^{\leq n}$, the truncation of the bicomplex obtained by replacing each column of $\mathscr{C}$ with its normalized chains, then applying the Alexander–Whitney map, as in the diagram below:

$$\begin{tikzcd} C_\bullet(X^q) \arrow[r, "\mathcal{AW}_q"] \arrow[d, two heads] & C_\bullet(X)^{\otimes q} = \mathscr{D}^q_\bullet \arrow[d, two heads] \\ N_\bullet(X^q) \arrow[r, "\mathcal{AW}_q"'] & N_\bullet(X)^{\otimes q} =: N\mathscr{D}^q_\bullet \end{tikzcd}$$

Notice that $N_\bullet(a)$ and $N_\bullet(b)$ are just the chain complexes with a $\mathbf{Z}$ term concentrated in degree zero, i.e. the identity object in the symmetric monoidal structure on $\mathsf{Ch}_+(\mathsf{Ab})$. Hence we have natural isomorphisms

$$\psi_a : N_\bullet(a)\otimes N_\bullet(X) \to N_\bullet(X)$$
$$\psi_b : N_\bullet(X)\otimes N_\bullet(b) \to N_\bullet(X)$$

which combine to give a natural isomorphism $\phi := \psi_a \otimes \mathrm{id}^{\otimes\bullet} \otimes \psi_b$:

$$\begin{aligned} \phi : \mathsf{Cobar}_{-p,q} = &\bigoplus_{n+b_1+\cdots+b_q+m=p} N_n(\{a\})\otimes N_{b_1}(X)\otimes\cdots\otimes N_{b_q}(X)\otimes N_m(\{b\}) \\ &\to \bigoplus_{b_1+\cdots+b_q=p} N_{b_1}(X)\otimes\cdots\otimes N_{b_q}(X) = N\mathscr{D}_{-p,q}. \end{aligned}$$

on the bigraded pieces. We now check that the differentials on Cobar and $N\mathscr{D}$ are compatible with ϕ. Because all differentials of $N_\bullet(a)$ and $N_\bullet(b)$ are zero, the horizontal cobar differential formula 4.2 yields

$$\begin{aligned} \phi(d_H^{\mathsf{Cobar}}(m\otimes c_1\otimes\cdots\otimes c_q\otimes n)) &= \phi\Big(\sum_{i=1}^{q}(-1)^{\sigma(c_i)} m\otimes c_1\otimes\cdots\otimes\partial_N(c_i)\otimes\cdots\otimes c_q\otimes n\Big) \\ &= \sum_{i=1}^{q}(-1)^{\sigma(c_i)}\psi_a(m\otimes c_1)\otimes\cdots\otimes\partial_N(c_i)\otimes\cdots\otimes\psi_b(c_q\otimes n) \\ &= d_{N\mathscr{D}}(\phi(m\otimes c_1\otimes\cdots\otimes c_q\otimes n)). \end{aligned}$$

The vertical differential in both cases is given by the alternating sum of coface maps d^i. We will show that $\phi \circ d^i_{\mathsf{Cobar}} = d^i_{N\mathscr{D}} \circ \phi$ for all i. For $i = 0$, the map $d^i_{N\mathscr{D}} : N\mathscr{D}^q_\bullet \to N\mathscr{D}^{q+1}_\bullet$ is induced by the following inclusion under the topological Alexander–Whitney map:

$$(x_1, \dots, x_q) \to (a, x_1, \dots, x_q).$$

Naturality of the Alexander–Whitney map implies that the following diagram commutes:

$$\begin{tikzcd}
N_\bullet(X^q) \arrow[r, "\cong"] \arrow[d, "\mathcal{AW}_q"'] \arrow[rr, bend left, "N_\bullet(d^0)"] & N_\bullet(\{a\} \times X^q) \arrow[r, "N_\bullet(\iota_a \times \mathrm{id}_{X^q})"] \arrow[d, "\mathcal{AW}_{\{a\},X^q}"] & N_\bullet(X^{q+1}) \arrow[d, "\mathcal{AW}_{q+1}"] \\
N_\bullet(X)^{\otimes q} \arrow[r, "\cong"] \arrow[rr, bend right, "d^0_{\mathscr{D}}"'] & N_\bullet(\{a\}) \otimes N_\bullet(X)^{\otimes q} \arrow[r, "N_\bullet(\iota_a) \otimes \mathrm{id}^{\otimes q}"] & N_\bullet(X)^{\otimes q+1}
\end{tikzcd}$$

where ι_a is the inclusion $\{a\} \hookrightarrow X$. But now we make the identifications

$$N_\bullet(\{a\}) \otimes N_\bullet(X)^{\otimes q} \cong N_\bullet(X)^{\otimes q}, \text{ and } N_\bullet(\iota_a) = 1_a$$

to conclude

$$(1_a \otimes \mathrm{id}^{\otimes q}) \circ \mathcal{AW}_q = \mathcal{AW}_{q+1} \circ N_\bullet(d^0)$$

which implies $d^0_{N\mathscr{D}} = 1_a \otimes \mathrm{id}^{\otimes q}$. Then again by naturality of the Alexander–Whitney map, we compute

$$\begin{aligned}
\phi(d^0_{\mathsf{Cobar}}(m \otimes c_1 \otimes \cdots \otimes c_q \otimes n)) &= \phi(\Delta^R(m) \otimes c_1 \otimes \cdots \otimes c_q \otimes n) \\
&= \phi\big(((\mathrm{id} \otimes \iota_a) \circ \mathcal{AW}(m)) \otimes c_1 \otimes \cdots \otimes c_q \otimes n\big) \\
&= \phi(m \otimes 1_a \otimes c_1 \otimes \cdots \otimes c_q \otimes n) \\
&= \psi_a(m \otimes 1_a) \otimes c_1 \otimes \cdots \otimes \psi_b(c_q \otimes n) \\
&= 1_a \otimes \psi_a(m \otimes c_1) \otimes \cdots \otimes \psi_b(c_q \otimes n) \\
&= (1_a \otimes \mathrm{id}^{\otimes q})\big(\psi_a(m \otimes c_1) \otimes \cdots \otimes \psi_b(c_q \otimes n)\big) \\
&= d^0_{N\mathscr{D}}(\phi(m \otimes c_1 \otimes \cdots \otimes c_q \otimes n))
\end{aligned}$$

The proofs for $d^1, \dots, d^q$ are analogous. Hence the restriction of ϕ to the truncated bicomplexes yields a quasi-isomorphism $\mathrm{Tot}\,\mathsf{Cobar}^{\leq n} \to \mathrm{Tot}\,N\mathscr{D}^{\leq n}$. We now have a zigzag of quasi-isomorphisms

$$\mathrm{Tot}\,\mathsf{Cobar}^{\leq n} \to \mathrm{Tot}\,N\mathscr{D}^{\leq n} \twoheadleftarrow \mathrm{Tot}\,\mathscr{D}^{\leq n}$$

from which the lemma follows. $\square$

Remark 6.5. We have not really used Adams' theorem. But what we have proven should suggest that the theorem is sort of true for the following reason. We have exhibited a quasi-isomorphism between the totalizations of the truncations of the two-sided cobar construction on $C_\bullet(X)$ and the totalization of the truncated $\mathscr{C}$, which is itself the bicomplex constructed by taking singular chains on the cosimplicial path space. So if we can somehow commute "chains on a totalization" and "totalization of chains," and take the limit of these truncations, we would have a version of Adams' theorem.

We now have all the requisite maps for the proof of the main theorem. Before we proceed, just a quick recapitulation of all the steps involved. At the core of the argument is the cosimplicial path space $P^\bullet_{a,b}X$. Taking singular chains on each cosimplex, we obtain a bicomplex, which is the same (up to truncating) to the bicomplex underlying the two-sided cobar construction on (b, X, a). In §5 we see that filtering this bicomplex in one direction gives rise to the fundamental group ring, and here we see that in the other direction we find the homology groups $H_n(X^n, X(n)^a_b)$.

Proof of Theorem 1.1. By Lemma 6.4, Lemma 6.3, and the discussion in the first subsection, we have series of isomorphisms

$$H_\bullet(\mathsf{Cobar}^{\leq n}) \xrightarrow{6.4} H_\bullet(\mathscr{D}^{\leq n}) \xrightarrow{6.3} H_\bullet(\mathscr{C}^{\leq n}) \to H_\bullet(\mathscr{N}^{\leq n}).$$

and consequently an isomorphism in degree zero

$$H_0(\mathsf{Cobar}^{\leq n}) \to H_0(\mathscr{N}^{\leq n}). \tag{6.2}$$

Now Corollary 5.8 yields an isomorphism

$$\mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{n+1} \to H_0(\mathsf{Cobar}^{\leq n}). \tag{6.3}$$

Combining Equations 6.2, 6.3 and applying Corollary 6.2, we obtain a surjection

$$\mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{n+1} \xrightarrow{\cong} H_0(\mathsf{Cobar}^{\leq n}) \xrightarrow{\cong} H_0(\mathscr{N}^{\leq n}) \twoheadrightarrow H_n(X^n, X(n)^a_b)$$

whose kernel is isomorphic to $\mathbf{Z}$ if $a = b$ and trivial otherwise, completing our journey. □

6.3. **Looking forward.** Let us conclude with a few words about why the authors from the introduction might have been interested in Theorem 1.1. This discussion will be very imprecise. Let X be a connected complex algebraic variety, i.e. something that locally looks like the zero-locus of a collection of homogenous polynomials in complex projective space. Then the rational cohomology of X has more structure than just that of a $\mathbf{Q}$-vector space; they also gain so-called *mixed Hodge structures*, as proven by Deligne. Using our theorem (or rather its rational dual), one can transfer these structures onto the groups $\mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^n \otimes \mathbf{Q}$. For example, fixing $x \in X$, path concatenation induces a bilinear map

$$\mathcal{I}/\mathcal{I}^{n+1} \times \mathcal{I}/\mathcal{I}^{n+1} \to \mathcal{I}^2/\mathcal{I}^{n+2}.$$

By Theorem 1.1, this corresponds to a bilinear map

$$H_n(X^n, X(n)^x_x) \times H_n(X^n, X(n)^x_x) \to H_{n+1}(X^{n+1}, X(n+1)^x_x).$$

To see how this correspondence is geometrically implemented, we refer the reader to [Loo25], in which Looijenga gives a formula for the map $\mathbf{Z}\pi_X(a,b) \to H_n(X^n, X(n)^a_b)$. As the theorem appears in [DG05], it is a stepping stone toward the theory of *motives*, which is far beyond the scope of both this thesis and our understanding, signaling that it is time to stop.

References

[Ada56] J. F. Adams. On the cobar construction. *Proceedings of the National Academy of Sciences of the United States of America*, 42(7):409–412, 1956.

[Bro59] Edgar H. Brown. Twisted tensor products, i. *The Annals of Mathematics*, 69(1):223, January 1959.

[Bro82] Kenneth S. Brown. *Cohomology of Groups*. Springer New York, 1982.

[Che73] Kuo-Tsai Chen. Iterated integrals of differential forms and loop space homology. *The Annals of Mathematics*, 97(2):217, March 1973.

[DG05] P Deligne and A Goncharov. Groupes fondamentaux motiviques de tate mixte. *Annales Scientifiques de l'École Normale Supérieure*, 38(1):1–56, January 2005.

[EML53] Samuel Eilenberg and Saunders Mac Lane. On the groups $H(\Pi, n)$, I. *The Annals of Mathematics*, 58(1):55, July 1953.

[EZ53] Samuel Eilenberg and J. A. Zilber. On products of complexes. *American Journal of Mathematics*, 75(1):200, January 1953.

[Gad23] Nir Gadish. Letter-braiding: bridging combinatorial group theory and topology, 2023.

[GJ09] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Birkhäuser Basel, 2009.

[Goo] Tom (https://mathoverflow.net/users/6666/tom-goodwillie) Goodwillie. What is the cohomology of this complex? MathOverflow. URL:https://mathoverflow.net/q/66045 (version: 2011-05-26).

[Hat00] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2000.

[JR79] S. A. Joni and G.-C. Rota. Coalgebras and bialgebras in combinatorics. *Studies in Applied Mathematics*, 61(2):93–139, October 1979.

[Loo25] Eduard Looijenga. On the motivic description of truncated fundamental group rings. *Journal of Topology and Analysis*, page 1–13, May 2025.

[Lur26] Jacob Lurie. Kerodon. `https://kerodon.net`, 2026.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Springer Berlin Heidelberg, 2012.

[May99] J Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2 edition, September 1999.

[Mil67] R. James Milgram. The bar construction and abelian h-spaces. *Illinois Journal of Mathematics*, 11(2), June 1967.

[MZZ25] J.P. May, Ruoqi Zhang, and Foling Zou. Unital operads, monoids, monads, and bar constructions. *Advances in Mathematics*, 461:110065, February 2025.

[Poi95] Henri Poincaré. Analysis situs. *Journal de l'École Polytechnique*, 2(1):1–123, 1895.

[Riv22] Manuel Rivera. Adams' cobar construction revisited. *Documenta Mathematica*, 27:1213–1223, 2022.

[Ser80] Jean-Pierre Serre. *Trees*. Springer Berlin Heidelberg, 1980.

[Sta75] John R. Stallings. *Quotients of the powers of the augmentation ideal in a group ring*, page 101–118. Princeton University Press, December 1975.

[Sta26] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2026.

[Wei95] Charles A Weibel. *Cambridge studies in advanced mathematics: An introduction to homological algebra series number 38*. Cambridge University Press, Cambridge, England, October 1995.

[Yua] Qiaochu (https://mathoverflow.net/users/290/qiaochu-yuan) Yuan. What is a coalgebra intuitively? MathOverflow. URL:https://mathoverflow.net/q/76529 (version: 2011-09-27).

[Zha19] Ruoqi Zhang. Two-sided bar construction. `http://math.uchicago.edu/~may/REU2019/REUPapers/Zhang,Ruoqi(Rachel).pdf`, 2019. REU paper.